



Computational Complexity of the Landscape I

Citation

Denef, Frederik and Michael R. Douglas. 2007. Computational complexity of the landscape I. *Annals of Physics* 322(5): 1096-1142.

Published Version

<http://dx.doi.org/10.1016/j.aop.2006.07.013>

Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:3138325>

Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA>

Share Your Story

The Harvard community has made this article openly available.
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

Computational complexity of the landscape I

Frederik Denef^{1,2} and Michael R. Douglas^{1,3}

¹ *NHETC and Department of Physics and Astronomy, Rutgers University,
Piscataway, NJ 08855-0849, USA*

² *Instituut voor Theoretische Fysica, KU Leuven,
Celestijnenlaan 200D, B-3001 Leuven, Belgium*

³ *I.H.E.S.,
Le Bois-Marie, Bures-sur-Yvette, 91440 France*

`frederik.denef@fys.kuleuven.be, mrd@physics.rutgers.edu`

ABSTRACT: We study the computational complexity of the physical problem of finding vacua of string theory which agree with data, such as the cosmological constant, and show that such problems are typically NP hard. In particular, we prove that in the Bousso-Polchinski model, the problem is NP complete. We discuss the issues this raises and the possibility that, even if we were to find compelling evidence that some vacuum of string theory describes our universe, we might never be able to find that vacuum explicitly.

In a companion paper, we apply this point of view to the question of how early cosmology might select a vacuum.

Contents

1. Introduction	2
2. The cosmological constant	5
2.1 Theoretical approaches to the cosmological constant problem	6
2.2 Landscape models	8
3. Computational complexity of finding flux vacua	11
3.1 The Bousso-Polchinski model	11
3.2 A toy landscape model	12
3.3 A brief introduction to complexity theory	13
3.4 Approximate algorithms, and physical approaches	16
3.5 Computational complexity of Bousso-Polchinski	17
3.6 Other lattice problems	18
3.7 F-theory flux vacua and other combinatorial problems in string theory	19
3.8 Beyond toy models	23
4. Related computational problems in physics	24
4.1 Spin glasses	24
4.2 Physically inspired methods for finding the ground state	26
4.3 Reformulation of other NP-complete problems as physical systems	26
4.4 Protein landscapes	27
4.5 Quantum computation	28
5. Problems harder than NP	30
5.1 Sharp selection principles based on extremalization	30
5.2 Even more difficult problems	34
5.3 Anthropic computing	35
5.4 Advice	37
6. Practical consequences	38
7. Conclusions	42
A. A simple pseudo-polynomial toy landscape solving algorithm	44
B. NP-completeness of subset sum	45
C. NP-completeness of 0/1-promise version of subset sum	46

1. Introduction

Since about 1985 string theory has been the leading candidate for a unified theory describing quantum gravity, the Standard Model, and all the rest of fundamental physics. At present there is no compelling evidence that the theory describes our universe, as testing its signature predictions (such as excitation states of the string) requires far more energy than will be available in foreseeable experiments, while quantum effects of gravity are guaranteed to be significant only in extreme situations (the endpoint of black hole evaporation, and the pre-inflationary era of the universe) which so far seem unobservable. Still, the degree of success which has been achieved, along with the beauty of the theory and the lack of equally successful competitors, has led many to provisionally accept it and look for ways to get evidence for or against the theory.

Much ingenuity has been devoted to this problem, and many tests have been proposed, which might someday provide such evidence, or else refute the theory. One general approach is to suggest as yet undiscovered physics which could arise from string/M theory and not from conventional four dimensional field theories. Examples would be observable consequences of the extra dimensions [11, 97], or in a less dramatic vein, cosmic strings with unusual properties [94].

Such discoveries would be revolutionary. On the other hand the failure to make such a discovery would generally not be considered evidence against string theory, as we believe there are perfectly consistent compactifications with small extra dimensions, no cosmic strings, and so on. Similarly, while we might some day discover phenomena which obviously cannot be reproduced by string theory, such as CPT violation, at present we have no evidence pointing in this direction.

Even if, as is presently the case, all observable physics can be well modeled by four dimensional effective field theories, it may still be possible to test string/M theory, by following the strategy of classifying all consistent compactifications and checking the predictions of each one against the data. If none work, we have falsified the theory, while if some work, they will make additional concrete and testable predictions. Conversely, if this turned out to be impossible, even in principle, we might have to drastically re-evaluate our approach towards fundamental physics.

While string/M theory as yet has no complete and precise definition, making definitive statements premature, as in [52], we will extrapolate the present evidence in an attempt to make preliminary statements which could guide future work on these questions. Thus, in this work, we will consider various ingredients in the picture of string/M theory compactification popularly referred to as the “landscape” [105], and try to address the following question.

Suppose the heart of the problem really is to systematically search through candidate string/M theory vacua, and identify those which fit present data. This includes the Standard Model, and the generally accepted hypothesis that the accelerated expansion of the universe is explained by a small positive value for the cosmological constant. This problem can be expressed as that of reproducing a particular low energy effective field theory (or EFT). Suppose further that for each vacuum (choice of compactification manifold, aux-

iliary information such as bundles, branes and fluxes, and choice of vacuum expectation values of the fields), we could compute the resulting EFT exactly, in a way similar to the approximate computations made in present works. Could we then find the subset of the vacua which fit the data? Since the “anthropic” solution to the cosmological constant problem requires the existence of large numbers of vacua, $N_{vac} > 10^{120}$ or so, doing this by brute force search is clearly infeasible. But might there be some better way to organize the search, a clever algorithm which nevertheless makes this possible?

This is a question in computational complexity theory, the study of fundamental limitations to the tractability of well posed computational problems. Many of the problems which arise in scientific computation, such as the solution of PDE’s to within a specified accuracy, are tractable, meaning that while larger problems take longer to solve, the required time grows as a low power of the problem size. For example, for numerically solving a discretized PDE, the time grows as the number of lattice sites.

Now we believe the problem at hand can be described fairly concisely, and in this sense the problem size is small. The data required to uniquely specify the Standard Model is 19 real numbers, most to a few decimal places of precision, and some discrete data (Lie group representations). Although one is not known, we believe there exists some “equation” or other mathematically precise definition of string theory, which should be fairly concise, and this is certainly true of the approximate versions of the problem we know presently. Thus the observation of the last paragraph would seem promising, suggesting that given the right equations and clever algorithms, the problem can be solved.

On the other hand, for many problems which arise naturally in computer science, all known algorithms take a time which in the worst case grows exponentially in the size of the problem, say the number of variables. Examples include the traveling salesman problem, and the satisfiability problem, of showing that a system of Boolean expressions is not self-contradictory (in other words there is an assignment of truth values to variables which satisfies every expression). This rapid growth in complexity means that bringing additional computing power to bear on the problem is of limited value, and such problems often become intractable already for moderate input size.

In the early 1970’s, these observations were made precise in the definition of complexity classes of problems [64, 93, 98, 40, 14]. Problems which are solvable in polynomial time are referred to as problem class **P**, and it was shown that this definition is invariant under simple changes of the underlying computational model (say, replacing a Turing machine program by one written in some other programming language). On the other hand, while no polynomial-time algorithm is known for the satisfiability problem, a proposed solution (assignment of truth values to variables) can be checked in polynomial time. Such problems are referred to as in class **NP** (non-deterministic polynomial). These are by no means the most difficult problems, as in others it is not even possible to check a solution in polynomial time, but include many intractable problems which arise in practice. We continue our introduction to complexity theory below, and define the ideas of **NP**-complete, **NP**-hard and other classes which are widely believed to be intractable.

In this work, we observe that the class of problems which must be solved in identifying candidate vacua in the string landscape are **NP**-hard. The basic example is the problem

of finding a vacuum with cosmological constant of order 10^{-120} in natural units. While in suitable cases it is not hard to argue statistically that such vacua are expected to exist, proving this by finding explicit examples may turn out to be intractable. Indeed the intractability of similar problems is well known in the contexts of the statistical mechanics of spin glasses, in protein folding, and in other fields in which landscapes of high dimension naturally appear. We came to suspect the NP-hardness of the string theory problem in the course of a computer search for flux vacua reported in the work [45], and the possibility has also been suggested independently in other works enumerating brane constructions [28]. Also, a suggestion similar to that in section 5 was made by Smolin in [103], based on the difficulty of finding the global minimum of a generic potential energy function.

In section 3, we prove our claim for the simplified Bousso-Polchinski model of the landscape, and explain why we expect it for all of the more refined models studied to date (see [15, 44, 46] and many other physics works). This does not necessarily imply that string theory is not testable, just as the NP-hardness of the ground state problem for spin glasses does not mean that the theory of spin glasses is not testable, but clearly this point deserves serious consideration.

More interestingly perhaps from a physical point of view, these observations lead to a paradox, analogous to one posed in the context of protein folding [88]. Namely, if it is so difficult for us to find candidate string theory vacua, then how did the universe do it? As has been pointed out by various authors ([4, 119] and references there), no known physical model of computation appears to be able to solve NP-hard problems in polynomial time. On the other hand, according to standard cosmology, the universe settled into its present minimum within the first few seconds after the end of inflation, as is suggested by the correct predictions of element abundances from models of nucleosynthesis. This would seem far too little time for the universe to search through the candidates in any conventional sense.

We will address this question at length in a companion paper [48], but we set the stage in this paper by broadening our discussion, beginning in section 4 with a survey of other areas in physics where similar questions arise. We then explain in section 5 how simple prescriptions for “vacuum selection principles” arising from quantum cosmology can actually lead to computational problems which are far more difficult than the simple NP-hardness of fitting the data. This will allow us to survey various aspects of complexity theory which will be useful in the sequel, particularly Aaronson’s notion of quantum computation with postselection [3].

In section 6 we make general comments on the somewhat more down to earth (?) question of how to test string theory in light of these remarks, and discuss a statistical approach to the problems they create. We conclude by stating a speculative and paradoxical outcome for fundamental physics which these arguments might suggest.

We should say that while we have included some background material with the aim of making the main points of this paper clear for non-physicists, at various points we have assumed a fair amount of standard particle physics background. We plan to provide a more pedagogical account of these issues elsewhere, intended for computer scientists and other non-physicists.

2. The cosmological constant

Since this is rather central to the physical motivation, let us briefly review the cosmological constant problem for the benefit of non-physicist readers. More details and more references can be found e.g. in [113, 35].

Soon after Einstein proposed his theory of general relativity, he and others began to explore cosmological models. While it seemed natural to postulate that the universe is static and unchanging, it soon emerged that his equations did not allow this possibility, instead predicting that the universe would expand or contract with time. The reason was simply that the universe contains matter, whose gravitational attraction leads to time dependence. To fix this problem, Einstein added an additional term, the so-called cosmological constant term or Λ , corresponding to an assumed constant energy density of the vacuum. This term can be chosen to compensate the effect of matter so that (fine-tuned) static solutions exist.

However, the redshift of galaxies discovered by Hubble in 1929 implies that the universe is not static but instead expands, in a way which was then well modeled by taking $\Lambda = 0$. Of course observations could constrain $\Lambda = 0$ only to some accuracy, and the possibility remained open that better observations might imply that $\Lambda \neq 0$.

Now, from the point of view of quantum theory, the energy density of the vacuum is not naturally zero, as quantum effects (vacuum fluctuations) lead to contributions to the vacuum energy. In some cases, these are experimentally measurable, such as the Casimir energy between conducting plates. Thus, there is no good reason to forbid them in general; one might expect a non-zero Λ of quantum origin.

To summarize, although Einstein’s original motivation for introducing Λ was not valid, other good motivations replaced it. In Weinberg’s words [114], “Einstein’s real mistake was that he thought it was a mistake,” and the problem of determining the value of Λ , both observationally and theoretically, has attracted much interest.

On the experimental side, astronomical observations long suggested that the average total energy density in the universe was comparable to the total density of visible matter, around $\rho \sim 10^{-30} \text{g cm}^{-3}$. Gradually, evidence accumulated for non-visible or “dark” matter as well, in fact making up a larger fraction than the visible density. All this was still compatible with $\Lambda = 0$, however.

The situation changed in the late 1990’s due to the accumulation of new observational evidence such as type Ia supernovae, microwave background anisotropies and dynamical matter measurements. At present, it is widely believed that this evidence requires our universe to contain a “dark energy,” with density around $\rho_\Lambda \sim 10^{-29} \text{g cm}^{-3}$ at the present epoch.

The simplest model of dark energy is a nonzero cosmological constant. As we discuss next, obtaining such a small nonzero value for the cosmological constant from theory is a notoriously difficult problem. Before moving on, we should say that one can also hypothesize alternative models for the dark energy, say with scalar fields which are varying with time at the present epoch. Without going into details, all such models involve comparable small nonzero numbers, which are typically even more difficult to explain theoretically. At present there is no data significantly favoring the other possibilities, while they are more

complicated to discuss, so we will restrict attention to the cosmological constant hypothesis in the following. Similar considerations would apply to all the other generally accepted models that we know about.

2.1 Theoretical approaches to the cosmological constant problem

We now go beyond models which describe the cosmological constant, and discuss explaining it within a more complete theoretical framework. Here, one long had the problem that direct computation of Λ was not possible, because the known field theories of quantum gravity are nonrenormalizable. However by analogy with better understood renormalizable field theories, one expects that in any such computation, Λ would be the sum of two terms. One is a term Λ_q arising from quantum effects and of order the Planck energy density, characteristic of quantum gravity. The other is a classical, “bare” cosmological constant Λ_0 , which in a field theory framework is freely adjustable.

Because of this adjustable term, one had no strong theoretical arguments favoring a particular value, but two alternatives were generally held out. One was that the order of magnitude of Λ would be set by the expected order of magnitude of the quantum term Λ_q . Now, the fundamental scale in quantum gravity is the the Planck energy ¹ density, M_{pl}^4 , so one might hypothesize that $\Lambda \sim \Lambda_q \sim M_{pl}^4$. This is huge, of order 10^{102} J/mm³ = 10^{85} kg/mm³, i.e. about 10^{55} solar masses in each volume unit the size of a grain of sand, and obviously in conflict with observation.

String theory is believed to provide a well-defined theory of quantum gravity with no free parameters, and thus one expects to be able to compute the value of Λ . While difficult, there have been efforts to do this in simplified toy models. To the extent that one gets definite results, so far these are consistent with the assumption that string theory is not fundamentally different from other quantum theories for the questions at hand, in that basic concepts such as vacuum energy, quantum corrections, effective potential and so forth have meaning, are calculable from a microscopic definition in a similar way to renormalizable quantum field theories, and behave much as they do there and in semiclassical quantum gravity. One can certainly question this idea [21], but since it enters at a very early stage into all present attempts to make detailed contact between string theory and the real world, major revisions of our understanding at this level would require restarting the entire theoretical discussion from scratch. We see no compelling reason to do this, and proceed to follow this widely held assumption.

In some quantum theories, especially supersymmetric theories, the vacuum energy is far smaller than naive expectations, due to cancellations. Now there is no reason to expect this for the well established Standard Model, but it might apply to a hypothetical extension of the Standard Model which postulates new fields and new contributions to the vacuum energy at some new fundamental energy scale M_f , which would set the scale of Λ_q . Experimental bounds on new particles and forces require roughly $M_f \geq 1$ TeV, and

¹We work in units with $c = \hbar = 1$, and define the Planck mass to be $M_{pl} \equiv (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV, where G is Newton’s constant. Λ will denote the actual energy density, and not the energy density divided by M_{pl}^2 as is often done in the cosmology literature. A useful online source for energy unit conversions and constants of nature is [57].

even if this bound were saturated, the vacuum energy would still be one Earth mass per mm^3 .

We do not and could not live in such a universe. If $\Lambda = M_{pl}^4$, the universe would have a Planck scale curvature radius and expand exponentially with a Planck scale time constant. If $\Lambda = (1 \text{ TeV})^4$, it would still have a sub-millimeter scale curvature radius and inflate exponentially on a time scale of less than a picosecond. For negative cosmological constants of this order the universe would re-collapse into a Big Crunch on these ultrashort time scales. Thus, the simple fact of our own existence requires Λ to be extremely small.

This led many to the second, alternative hypothesis, which was that Λ should be exactly zero, in other words the adjustable term $\Lambda_0 = -\Lambda_q$, for some deep theoretical reason. Now in analogous problems, there often are arguments which favor the parameter value zero. For example, if a nonzero value for a parameter α breaks a symmetry, one can argue that quantum corrections will themselves be proportional to α , making $\alpha = 0$ a self-consistent choice. While no fully convincing argument of this type for $\Lambda = 0$ has ever been found, the simplicity of this hypothesis makes it hard to ignore. Thus it was that theorists found themselves in a way repeating “Einstein’s mistake” in reverse, unable to predict with any confidence that $\Lambda \neq 0$ was a serious possibility until observation made it apparent.

Now there was one pre-1990’s theoretical idea which did lead to such small values in a fairly natural way. This was that the fundamental theory should contain a large number N_{vac} of vacuum configurations, realizing different effective laws of physics, and in particular with different values of Λ . As we will see later, this is easy to achieve in field theory, in many ways, and appears to be true in string theory. Given an appropriate distribution of Λ values, it then becomes statistically likely that a vacuum with the small observed value of Λ exists.

To illustrate, suppose the number distribution of Λ among vacua were roughly uniform, meaning that the number of vacua with Λ in an interval $\Lambda \in (a, b)$ were roughly

$$N_{vac}(a \leq \Lambda \leq b) \sim N_{vac} \frac{b - a}{2M_{pl}^4},$$

for any $|a|, |b| \leq M_{pl}^4$. If so, the claim that there exists a vacuum with $\Lambda \sim M_{pl}^4/N_{vac}$ becomes statistically likely, and this might be considered sufficient evidence to take the claim that such a theory can solve the cosmological constant problem seriously. Of course, there are clearly pitfalls to guard against here, such as the possibility that the distribution has unusual structure near $\Lambda = 0$, correlations with other observables and so forth, but keeping these in mind, let us proceed.

This argument does not yet explain why we find ourselves in a vacuum with a small value of Λ . One might expect such a question to be explained by the dynamics of early cosmology, and thus try to identify a dynamical mechanism which leads with high probability to a vacuum with an extremely low or perhaps even the lowest positive Λ . We will discuss this idea below and in [48], but for now let us just say that this appears problematic.

A different approach is to claim that all of the possible vacua “exist” in some sense, and that at least part of the structure of the vacuum we observe is simply environmen-

tally selected: we can only find ourselves in a vacuum where the conditions are such that something like an observer asking these questions can exist. This general idea is known as the “anthropic principle” [25]. Since as we mentioned, the simple fact of our own existence requires Λ to be extremely small, this principle would seem highly relevant here.

Various objections have been raised to the anthropic principle. While at first it may seem tautological, it clearly does have content in the context of a physical theory which predicts many possible candidate laws and structures for our universe, as it often provides a very simple answer to the question “why do we not find ourselves in vacuum X.” There are more serious objections. It is unimaginably difficult to characterize the most general conditions which might allow for observers who can ask the questions we are addressing. Even restricting attention to simple necessary conditions, analyzing their dependence on the many fundamental parameters is complicated. Finally, if there are many “anthropically allowed” vacua, it does not lead to any clear preference among them. But keeping these caveats in mind, the principle has led to interesting claims.

The original anthropic argument bearing on the cosmological constant took as the necessary condition the requirement that some sort of structure such as galaxies, could form from an initially smooth post Big Bang universe by gravitational clumping of matter. This puts a bound on the cosmological constant because if Λ is significantly bigger than the matter density at the time when gravitational clumping starts, the acceleration of expansion driven by Λ will outpace the clumping process, and the universe ends up as a dilute, cold gas. Assuming all other parameters fixed, Weinberg [112] computed in 1987 that this requires $\Lambda < 400\rho_0$, where ρ_0 is the present matter density. A (negative) lower bound is obtained by requiring the universe to survive for long enough to allow for some form of life to evolve, before re-collapsing into a Big Crunch. Again assuming all other parameters fixed, this gives a bound $\Lambda > -\rho_0$. Together this gives the allowed window

$$-10^{-120}M_p^4 < \Lambda < 10^{-118}M_p^4. \quad (2.1)$$

Although there are a number of assumptions that went into this computation, most notably fixing the amplitude of primordial density fluctuations, and although the window is still two orders of magnitude wider than the observed value of Λ , this is by far the most successful (and simple) computation of Λ produced in any theoretical framework to date. Indeed, it might be regarded as a prediction, as it came well before the evidence.

While this argument is clearly important, as we discussed we now have direct evidence for non-zero dark energy, and thus part of “fitting the data” is to reproduce this fact. Despite many attempts, at present the only theoretical approach in which we can convincingly argue that this can be done is the statistical argument we discussed.

2.2 Landscape models

A “vacuum” is a candidate ground state of a physical theory. It should be either time independent, *i.e.* stable, or extremely long lived compared to the physical processes under consideration, *i.e.* metastable. Since we seek a theory which can describe all physics through all observed time, its average lifetime must far exceed the current age of the universe, of order 10^{10} years.

In well understood theories, to a good approximation stability is determined by classical considerations involving an energy functional called the effective potential (part of the EFT mentioned in the introduction). Both stability and metastability require that the energy increases under any small variations of the configuration. In quantum theory, one can also have instability caused by tunneling events to lower energy configurations. Thus, acceptable metastability requires that the barriers to tunneling are so high that tunneling rates are negligible on scales far exceeding the current age of the universe.

A simple model for the vacuum energy and its dependence on the configuration is to describe the configuration as a “scalar field,” a map ϕ from space-time into some manifold \mathcal{C} . The vacuum energy functional E is then determined by a real valued function V on \mathcal{C} , the potential. It is given by an integral over all space, which if the derivatives of the fields are small has an expansion

$$E = \int d^3x \sqrt{g} V(\phi(x)) + \mathcal{O}\left(\left(\frac{\partial\phi}{\partial x}\right)^2\right).$$

In this case, a vacuum is a constant field configuration $\phi = \phi_0$ with $\partial\phi/\partial x = 0$, which is a local minimum, *i.e.* a critical point with positive definite Hessian,

$$\frac{\partial V}{\partial\phi} = 0; \quad \frac{\partial^2 V}{\partial\phi\partial\phi} > 0.$$

The value at the minimum $\Lambda = V(\phi_0)$ is the energy of the vacuum, in other words the cosmological constant in that vacuum.

Such a model already suffices to realize the scenario we described, in which there are many vacua realizing widely differing values of Λ . The simplest models of this type, which were also the first to be proposed [5], simply take $\mathcal{C} = \mathbb{R}$ and a “washboard” potential such as

$$V(\phi) = a\phi - b \cos 2\pi\phi. \tag{2.2}$$

For $a \ll b$, this potential has many minima, with equally spaced cosmological constants $\Lambda = an - b$ for all $n \in \mathbb{Z}$. Thus we simply need $a \sim 10^{-120} M_{pl}^4$ for some vacuum to realize the observed Λ . There are many variations on this construction, which solve the problem with regular potentials depending on a few fields.

Instead of continuously adjustable fields ϕ and a potential with isolated minima, another possibility is to find models with discrete choices. One systematic discrete choice, which will be exploited in the Bousso-Polchinski model below, is that of “flux.” This is a postulated magnetic field (or generalized “ p -form magnetic field”) in the extra dimensions. As in electrodynamics, such a magnetic field will contribute to the potential energy as $V \sim B^2$. But whereas ordinary magnetic field breaks rotational symmetry, this one only breaks symmetries in the extra dimensions, so such a configuration is still a “vacuum.” One might ask why it is stable, as a localized configuration of ordinary magnetic field is not. This is because of Dirac’s quantization condition for magnetic charge. On a topologically nontrivial manifold M , this generalizes to the statement that, after expressing the magnetic field in appropriate units, its integral over any homology cycle in M is an integer, and thus the field cannot vary continuously or decay (classically).

While this is different microscopically from a scalar field, its effect on observed physics is not very different, and roughly corresponds to taking $\phi \in \mathbb{Z}^n$ in a potential like the ones we just postulated.² One could again postulate the potential Eq. (2.2) to get a candidate solution to the cosmological constant problem, or even derive it from a microscopic model as in [30, 31].

The models we just discussed have the serious flaw that the problem of obtaining the small coefficient $a \sim 10^{-120} M_{pl}^4$ in Eq. (2.2) is just as hard as solving the original cosmological constant problem. A way around this, suggested in [18], is to take $\mathcal{C} = \mathbb{R}^2$ or \mathbb{Z}^2 , with coordinates ϕ^1, ϕ^2 , and

$$V(\phi) = a_1 \phi^1 - b_1 \cos 2\pi \phi^1 + a_2 \phi^2 - b_2 \cos 2\pi \phi^2 + b_1 + b_2 \quad (2.3)$$

If the ratio a_1/a_2 is irrational, then the set of possible cosmological constants $\Lambda = a_1 n_1 + a_2 n_2$ will be dense in \mathbb{R} , without the need to postulate small numbers. To get a finite but large set, one can bound the range of ϕ^1 and ϕ^2 .

While these models would not lead to the complexity issues we are about to discuss, they are also drastic oversimplifications of the real models which arise from string theory compactification. In practice, the potential is never as simple as Eq. (2.2) or Eq. (2.3); indeed the problem of finding any controlled description of the potential which is valid over the large range of ϕ values required in either of these models is difficult. While the discussion rapidly gets technical, in the better understood models arising from string theory, such as [68, 15, 46, 44], one finds that in models with few fields, the required large number of quasi-realistic minima is not present.

But this is not the only way we could imagine finding a large set (or “discretuum” [29]) of values of Λ . Another general possibility is for \mathcal{C} to be a space of high dimension, with V a fairly generic function on \mathcal{C} . In string theory compactification, $\dim \mathcal{C}$ is typically determined by a topological invariant of the compactification manifold, say a Betti number, and can take values ranging up to 1000 or more. As is familiar in the study of optimization problems, and as we are about to discuss at great length, generic functions on high dimensional spaces typically have a number of minima which grows exponentially in the dimension, and realize many different values at the local minima.

Potentials of this type, depending on many (continuous or discrete) variables, appear naturally in many areas of physics and mathematical biology, and are often referred to as “landscapes.”³ As we discuss shortly, they also appear naturally in string theory, which has led to the name “string theory landscape” [105].

It should be clear that the idea of a landscape is *a priori* completely independent of the idea of anthropic or environmental selection principles: even in the presence of a landscape,

²One might think that the two types of model differ in whether vacua with different values of ϕ are connected by physical time evolution, as the flux cannot change classically. However, in both cases ϕ can change via tunneling events.

³The term first appeared in the phrase “fitness landscape” in evolutionary biology [118]. While we are not historians, from perusal of reviews such as [110, 89], it appears that systematic consideration of energy landscapes in physics and physical chemistry began in the 1970’s, with the term “landscape” coming into common though occasional use in the mid-1980’s.

one could in principle imagine a strong dynamical selection mechanism producing the apparent fine-tunings we observe. Thus we will try to carefully distinguish these ideas in the following.

3. Computational complexity of finding flux vacua

We begin by reviewing the Bousso-Polchinski (BP) model [29], to set out the problem. We then turn to issues of computational complexity in this context. As a warm-up example we consider a toy model of Arkani-Hamed, Dimopoulos and Kachru [13], for which the analysis is very easy. We then proceed to the detailed analysis of the complexity of the BP problem, and briefly discuss related problems and more sophisticated models of the string theory landscape.

3.1 The Bousso-Polchinski model

Stripped down to its essence, the Bousso-Polchinski model consists of a set of K quantized fluxes given by a vector $N \in \mathbb{Z}^K$, giving rise to a vacuum energy

$$\Lambda = \Lambda_0 + \sum_{ij} g_{ij} N^i N^j. \quad (3.1)$$

Here Λ_0 is some bare negative cosmological constant (which can be thought of as coming from orientifold, curvature and other flux-independent contributions), and g_{ij} some positive definite metric. The dimension K is determined by topological invariants of the compactification manifold; such as a Betti number; in examples this can range up to 1000 or so. In general g_{ij} and Λ_0 will depend on many scalar fields (moduli), which in turn are stabilized at minima of the vacuum energy function. To avoid having to deal with such a complicated coupled system, the BP model simply assumes the moduli to be frozen, and g_{ij} and Λ_0 to be given constants. The fully coupled system as arising in actual string compactifications was analyzed in [15, 46, 44], confirming the qualitative picture emerging from the BP model.

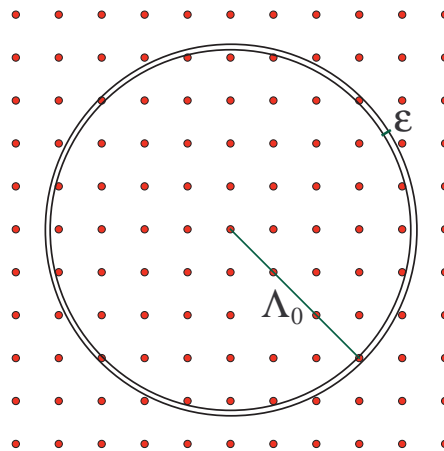


Figure 1: Scanning the cosmological constant in the Bousso-Polchinski model

A crucial observation of [29] was that for sufficiently large K and $|\Lambda_0|$, the cosmological constant Λ gets sufficiently finely scanned near zero to allow for extremely small values, including numbers of the order of the presently observed cosmological constant $\Lambda \sim 10^{-120} M_p^4$. The idea, sketched for $K = 2$ in figure 1, is that for large K and $|\Lambda_0|$, even a very thin shell of radius $\sqrt{-\Lambda_0}$ will contain some lattice points.

The basic estimate is as follows. A ball defined by $g_{ij}N^iN^j \leq L$ has Euclidean volume

$$\text{Vol}(L) = \frac{(\pi L)^{K/2}}{(K/2)!} \frac{1}{\sqrt{\det g}} \quad (3.2)$$

in N -space. For $\text{Vol}(L) \gg 1$, this is a good estimate for the number of lattice points contained in the ball. Similarly, a thin shell of radius squared $|\Lambda_0|$ and width ϵ has a volume

$$\delta\text{Vol} \approx \frac{K}{2|\Lambda_0|} \text{Vol}(|\Lambda_0|) \epsilon \quad (3.3)$$

and hence when this is much bigger than 1, we expect about δVol vacua (labeled by different values of N) with a cosmological constant of order ϵ .

Suppose however that we want to do better than these estimates and find out exactly which vacua have cosmological constant $\Lambda \in (0, \epsilon)$, or at least find out with certainty whether or not there is one in this interval.⁴ A related problem is to find the vacuum with the smallest positive cosmological constant. The first problem would be a maximally simplified model of finding string vacua in agreement with observational data in physical parameter space. The second kind of problem would naturally arise for instance if we grant an *a priori* probability distribution $P(\Lambda) \sim \exp(c/\Lambda)$ on the landscape of vacua, since such a distribution strongly peaks on the smallest positive value of Λ . This particular choice of $P(\Lambda)$ can be motivated in various cosmological frameworks; it will be discussed in detail in later sections.

Clearly, none of these problems are easy. But before we go on and quantify just *how* hard they are, we consider an even simpler model, for which the complexity analysis is more straightforward.

3.2 A toy landscape model

In [13], Arkani-Hamed, Dimopoulos and Kachru (ADK) proposed a field theory toy model for the landscape consisting of N scalars ϕ^i and a potential

$$V(\phi) = \sum_i V_i(\phi^i) \quad (3.4)$$

where each V_i has two minima, $V_i^- < V_i^+$. This model has 2^N vacua, with vacuum energies

$$\Lambda_m = \sum_i m_i(V_i^+ - V_i^-) + V_i^- = \sum_i m_i \Delta V_i + V_{\min} \quad (3.5)$$

⁴Of course, to make this a meaningful question, we need to know all given quantities to a sufficiently high precision. We assume this for now but will discuss this issue in more detail further on.

where $m_i \in \{0, 1\}$. The distribution of vacuum energies was studied in [13] in the large N approximation, in which the central limit theorem can be used to argue that this distribution will be a Gaussian.

Suppose however that we want to find the precise smallest positive value of Λ in this ensemble, or want to find out with certainty if there exists a vacuum with Λ within the range $(\Lambda_0, \Lambda_0 + \epsilon)$. The most naive algorithm to solve this problem would be to simply loop through all vacua. This would take 2^N steps, which becomes exponentially quickly infeasible for large N ; for $N = 50$, scanning one billion instances per second, this would take about two weeks. For $N = 400$, the minimal value to (naively at least) expect vacua with Λ of order 10^{120} , it would take 10^{104} years.

3.3 A brief introduction to complexity theory

Can the preceding algorithm be improved to give a solution in some time polynomial in N rather than exponential in N ? Questions like this are the subject of computational complexity theory, to which we now give a brief introduction.

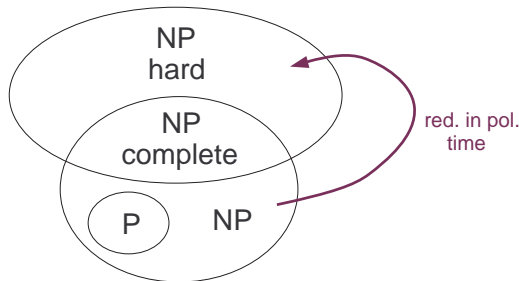


Figure 2: Basic complexity classes

The following basic complexity classes are relevant for the problem at hand:⁵

- **P** (Polynomial time): Decision (yes/no) problems solvable by an ordinary computer (or more formally a deterministic Turing machine) in a number of steps polynomial in the input size. For example the decision problem “Is $N = N_1 \cdot N_2$?” is in P, since the time needed to multiply two integers is polynomial in the number of digits. A highly nontrivial example of a polynomial time problem is primality testing [10].
- **NP** (Non-deterministic Polynomial time): Decision problems for which a candidate solution can be verified in polynomial time. More precisely, if the answer is yes, there should exist a proof of this verifiable in polynomial time. Typically an NP problem is of the form “Does there exist an x such that $p(x)$ is true?”, where “ $p(x)$ is true” can be verified in polynomial time for any given x . The problem of finding out whether or not there exists a vacuum in the ADK toy landscape with energy in a specified

⁵See [93, 98, 14] for a detailed explanation, and [38] for a huge up-to-date list of complexity classes. For a more elementary online introduction, see [117].

range is in NP, since a candidate example can be checked in time polynomial in the input size.⁶

Equivalently, these are the problems that can be solved in polynomial time on a *non-deterministic* Turing machine. Roughly speaking, the difference between a deterministic Turing machine and a non-deterministic one is that the former works like a standard computer, executing instructions sequentially in one chain, whereas the latter is allowed to branch off in a number of “copies” at each step, with each copy executing a different operation at that step. Hence, rather than a single computation path, we now have a computation tree. If the answer is yes, at least one of the computation paths should accept. If the answer is no, all computation paths should reject. For the toy landscape problem, the branches at level i of the tree could be taken to correspond to picking $m_i = 0$ resp. $m_i = 1$. Alternatively, one can think of the branching as flipping a coin. NP problems are then problems for which a polynomial time randomized algorithm exists that always outputs no when the answer is no, and at least in one possible run outputs yes when the answer is yes.

Obviously $P \subseteq NP$.

- **NP-hard:** Loosely speaking, this is the class of problems at least as hard as *any* NP problem. More precisely this means that any NP problem can be reduced to it in polynomial time, in a sense we will explain in an example shortly.
- **NP-complete:** An NP-hard problem that is in NP itself is called NP-complete. In this sense they are the hardest problems in the class NP. It is quite remarkable that natural NP-complete problems exist at all, but by now a large collection of such problems is known. One simple example is the subset-sum problem: given a finite list of integers and a target integer t , does there exist a subset that sums up to t ?

By definition, if *any* NP-hard problem can be solved in polynomial time, *all* NP problems can be solved in polynomial time, and therefore we would have $P=NP$. It is widely believed that this is not the case, although there is no proof, and the P versus NP problem is considered to be the biggest open question in theoretical computer science. So big in fact that it is one of the seven Clay Millennium Prize problems [36].

While we will not need complete and precise definitions of these concepts for our discussion, a point worth keeping in mind is the difference between decision problems, which take an input and produce a yes or no output, and problems which produce more general outputs. The classes we just described refer to decision problems. While similar definitions can be made for other problems (say for computing a function, we have FP, FNP and so on), to simplify our discussion we will stick to decision problems.

Thus, we pose the ADK toy landscape problem: does the set of vacuum energies Λ_m obtained from Eq. (3.5), contain a vacuum in the range $(\Lambda_0, \Lambda_0 + \epsilon)$?

The bad news is that this problem is already NP-complete. It is a special case of the classic *knapsack problem* (and closely related to the subset sum problem): given a list of

⁶We are assuming here that the energies V_i^\pm are given up to a fixed precision and represented as rational numbers. We will discuss precision issues further on.

items with cost c_i and value v_i , can a subset of items be selected such that the total value exceeds a given v_{\min} but the total cost remains below c_{\max} ? This problem is well known to be NP-complete [79, 64] and reduces to our landscape problem by taking

$$v_i = c_i = \Delta V_i, \quad v_{\min} = \Lambda_0 - V_{\min}, \quad c_{\max} = \Lambda_0 - V_{\min} + \epsilon.$$

This means that unless NP is the same as P, there is no polynomial time algorithm to solve our problem exactly for all possible instances.

While the proof of NP-completeness of the knapsack problem or the closely related subset sum problem is given in textbooks [64, 93], let us briefly summarize the approach. Suppose we want to prove NP-completeness of subset sum. First, we choose a known NP-complete problem. Then, we show that for any instance of that problem, we can construct a subset sum problem whose solution could be translated back into a solution of the original problem. This is referred to as a “reduction” of our original problem to subset sum, and the last point to check is that this reduction must be doable in a polynomial number of elementary operations.

A convenient choice for the known NP-complete problem is the satisfiability or *SAT* problem. In this problem, one is given a list of N Boolean variables, each taking the value true or false, and a list of clauses, each a Boolean OR in which a subset of the variables appear, either literally or negated. For example, labelling the variables x_i , a sample clause would be

$$x_2 \vee \bar{x}_7 \vee x_{22}$$

where the bar represents NOT, and \vee represents OR. The problem is to find an assignment of truth values to variables which satisfies all of the clauses. Since a proposed assignment can be checked in time proportional to the number of clauses, this problem is in NP. A closely related variant is the k -SAT problem, in which a definite number k of variables appear in each clause. One can show that any SAT problem can be (polynomially) mapped into a 3-SAT problem, by introducing new variables and subdividing clauses.

The SAT problem was shown to be NP-complete [40] as follows. We need to argue that, given a Turing machine program which verifies a solution to some problem in polynomial time, we can translate this verification into a SAT problem. This is done by considering a “tableau” which is simply the runtime history of a Turing machine computation, step by step. Now, any valid history will satisfy a set of logical relations between subsequent steps, whose simultaneous truth implies that some particular run of the program results in the machine deciding that the original problem has a solution. These relations give us a SAT problem with a polynomial number of clauses, whose solution amounts to the verification of a solution to our original problem. By feeding this to a hypothetical polynomial time SAT solver, we could solve the original decision problem in polynomial time.

The reduction of SAT (or 3-SAT) to subset sum is analogous although of course the details are different. Given a set of Boolean clauses, one needs to construct a set of numbers which will sum to a given number only if the 3-SAT problem has a satisfying assignment. By the trick of using “base b ” numbers of the form $N = \sum c_k b^k$, one can express the condition that many different (small) c_k ’s sum to specified numbers simultaneously, so we

just need to encode the conditions that each variable is either true or false, and each clause is true, into a set of subset conditions on independent c_k 's. In appendix B, we give a detailed proof based on this idea.

The upshot is that finding a general polynomial time subset sum solver would imply $P = NP$, which is generally believed to be false, and thus we believe that the answer to the question we began the subsection with is “no.”

3.4 Approximate algorithms, and physical approaches

The previous argument, while standard in computer science, is a bit intricate and quite different from the usual physical discussions of such problems. Now such phenomena are already known in other branches of physics, and we discuss these connections in section 4. But let us first discuss some possible ways around it, by relaxing the rather strong requirements we set out, or adopting more physical approaches.

First, while the argument we just gave shows clearly that the complexity of the problem grows with the number of fluxes, it does not directly refer to the size or accuracy of the energies we are considering. Indeed, in practice, we will usually not know the exact values of the V_i^\pm , even if they are computable in principle from a fundamental theory. For example, various quantum corrections may be quite hard to compute.⁷

Thus, let us assume we know the values of the V_i^\pm to a given accuracy of order δ . The problem we posed is then only sensible of course if $\delta < \epsilon/N$. Now for *fixed* values of δ and ΔV_i , the solution can be found (within the maximal precision determined by δ) in time polynomial in N . In appendix A, we give a simple algorithm solving the problem in $O(N(\sum_i \Delta V_i)/\delta)$ time. If an order ϵ error in the target energy range is allowed, we can assume $\delta = \epsilon/N$, so the required time is of order $N^2(\sum_i \Delta V_i)/\epsilon$. Hence if ϵ (or δ) is not much smaller than the ΔV_i , the problem becomes effectively tractable. This is an example of an NP-hard problem with a “fully polynomial time approximation scheme” (FPTAS). By no means are all NP-hard problems approximable in this strong sense, see e.g. [92, 84].

The catch is that if we want this toy landscape problem to model the problem of finding out whether there exist string vacua with cosmological constant of order $10^{-120} M_p^4$, and the typical value of ΔV_i is Planck scale, we should take $\sum_i \Delta V_i/\epsilon \sim 10^{120}$. Then, the algorithm of appendix A becomes effectively useless. This illustrates that it is also the accuracy to which we must solve the problem, as well as the large number of fluxes, which makes it difficult.

We can contemplate different, more “physical” algorithms to find vacua with minimal or small cosmological constant. For example, we could just start off a particle high up a hill of $V(\phi)$ and try to let it roll down to a vacuum with cosmological constant in $(0, \epsilon)$. Clearly for small ϵ this is not going to work: there are so many local minima at values of Λ well outside $(0, \epsilon)$ that we would quickly get stuck, and would almost never end up in the target interval. A similar conclusion holds for a discretized version where we stick to the actual vacua and define local jumps as flipping one m_i between 0 and 1. We could add

⁷These quantum corrections will typically also spoil the simple direct form of the potential, but we will ignore this for the sake of the argument.

thermal noise to the motion of the particle such that it gets kicked out of local minima after some time and continues its path down (together with some kick-back mechanism⁸ when V drops below zero, to keep the energy positive), and indeed such procedures can be shown to eventually do the job, but again only after exponentially long time. Such algorithms are nevertheless often very useful in practice, especially if one is satisfied with an approximate solution. We will discuss them in more detail in the next section.

This example illustrates a deep conceptual connection between **NP**-hardness and physics: when a certain problem is **NP**-hard, *any* “physical” way of solving it by rolling over some effective potential landscape or by some other local relaxation mechanism will come with exponentially many local minima, leading to exponential relaxation times to the lowest lying minima. This has immediate dynamical implications for real physical systems: even when there is a selection principle singling out a preferred state, such as minimal energy, if the problem that needs to be solved to satisfy this selection criterion is **NP**-hard, the system may well be unable to reach this preferred state on any reasonable time scale. This is true for instance for spin glasses. We will look at this and its consequences for fundamental physics in much more detail in subsequent sections.

3.5 Computational complexity of Bousso-Polchinski

We now return to the complexity analysis of the Bousso-Polchinski model. The problem we want to solve is the following: Does there exist an $N \in \mathbb{Z}^K$ such that

$$\Lambda_1 \leq \Lambda_0 + \sum_{ij} g_{ij} N^i N^j \leq \Lambda_2? \quad (3.6)$$

Here K , g_{ij} , and Λ_r are all part of the input (and $\Lambda_0 < 0$).

We prove that this problem is **NP**-complete by relating it to a version of the subset sum problem.⁹ As in appendix A, since we work at some finite precision, the values of Λ_1 , Λ_2 and g_{ij} can all be assumed to be integral after a suitable choice of energy units.

To show that this problem is **NP**-complete, we have to show that some known **NP**-complete problem can be reduced to it in polynomial time. The problem we choose for this is a version of subset sum with some extra information (a “promise”) about the input. The question we ask is, given a set of K positive integers $\{x_i\}$ and an integer t , do there exist $k_i \in \{0, 1\}$ such that $\sum_i k_i x_i = t$? The extra information is that the x_i are such that $\sum_i k_i x_i \neq t$ for any $k_i \in \mathbb{Z}^+$, unless all $k_i \in \{0, 1\}$. In other words, in this problem we are promised that the target t cannot be reached by summing with multiplicities. The **NP**-completeness of this promise problem can be derived from the **NP**-completeness of the standard subset sum problem. The proof is a bit technical, and we defer it to appendix C.

Building on this, the reduction to the Bousso-Polchinski problem is straightforward: just take $g_{ij} = x_i \delta_{ij}$, $\Lambda_1 = \Lambda_2 = t$. Thanks to the promise on the input, we know that the fluxes N_i can only take values 0 or ± 1 in any solution of this problem, and therefore

⁸Trying to minimize $V(\phi)^2$ instead will not work in this approach: V^2 has many codimension one loci of unwanted local minima, namely the zero set of V . This can be avoided however in the discretized version of the algorithm where we stick to the actual minima and sequentially flip components of m .

⁹We thank Dieter Van Melkebeek for suggesting this approach to us.

this instance of the BP problem becomes identical to the above version of the subset sum problem. This completes the proof.

The occurrence of exponentially many local minima in a local minimization approach to the problem can be seen as follows. Assume we take target values Λ_1 and Λ_2 to be very near 0, say within $\epsilon \ll g$, where g is the scale of g_{ij} . For simplicity we take $g_{ij} = g_i \delta_{ij}$. We can try to reach the target interval by starting at some large N and minimizing

$$|\Lambda| = \left| \sum_i g_i (N^i)^2 + \Lambda_0 \right|. \quad (3.7)$$

A single step in this procedure would consist of changing a random component N_k of N by one unit: $\Delta N_i = \pm \delta_{ki}$. This changes Λ by an amount

$$|\Delta \Lambda| = g_k |1 \pm 2N^k| > g_k.$$

The minimization procedure will thus quickly relax down $|\Lambda|$, but as soon as a value $|\Lambda| < g_k/2$ is reached, any subsequent elementary step will in fact *increase* $|\Lambda|$. That is, we are at a local minimum. At this point, if $\epsilon \ll \min_k g_k$, we are still very far from the target interval, and moreover there are clearly exponentially many such local minima. Again, we can try to further progress down by adding thermal noise, but as we come closer and closer to $\Lambda = 0$, finding still smaller values of $|\Lambda|$ (if they still exist) becomes increasingly hard, as the smaller values will get further and further away from each other. For the tiny values of ϵ we are interested in, it will thus take exponentially many jumps to get in the target range.

One could also consider the “algorithm” followed by cosmological relaxation mechanisms proposed in this context [30, 31, 29, 60]. We will analyze these and discuss the corresponding implications of computational complexity in detail in [48].

3.6 Other lattice problems

The Bousso-Polchinski problem is similar to well known lattice problems such as the *shortest lattice vector problem* (SVP): given a lattice in \mathbb{R}^N , find the shortest vector. It is less obvious that this problem¹⁰ is NP-complete. For example, unlike in the BP problem, a minimization algorithm would not need to explore increasingly further apart lattice points. And indeed in the diagonal g_{ij} case the problem is trivially in P, the minimum length being $\min_k g_k$. In fact, the NP-hardness of this problem was long an open question, but in 1998, Ajtai [8] proved it to be NP-hard under randomized reductions (which is slightly weaker than standard NP-hardness). Many approximation algorithms are known, most famously the LLL algorithm [86], which finds a short vector in polynomial time that is guaranteed to be at most a factor $2^{K/2}$ longer than the actual shortest vector. Various lower bounds on polynomial time approximability are known, see e.g. [77] for an overview.

Another well known NP-hard lattice problem is the *closest lattice vector problem*: given a lattice and a point in \mathbb{R}^K , find the lattice point closest to that point. There are many other hard lattice problems. In particular, a remarkable “0-1 law” conjecture by Ajtai [9]

¹⁰or more precisely its decision version; we will be a bit sloppy in our terminology in this section.

produces a huge set of NP-complete lattice problems. The conjecture roughly says that any polynomial time verifiable property of a lattice becomes generically true or generically false for random lattices in the large lattice dimension K limit. In other words, only properties which in the large K limit can be statistically excluded or statistically guaranteed can actually be possibly verified in polynomial time. Any property that would be somewhat restrictive but not too restrictive would automatically be intractable. An example is the question whether there is a lattice point in some given region of volume not much smaller or larger than the volume of a lattice cell. The probability that there is such a point remains bounded away from 0 and 1 when $K \rightarrow \infty$, so if the conjecture holds, answering this membership question is a problem not in P. This automatically includes SVP, CVP and BP. The conjecture actually also implies that $\text{NP} \neq \text{P}$, so there is not much hope of proving it any time soon.

In the previous subsection we considered the problem of trying to match the cosmological constant. One could consider different parameters, such as particle masses, Yukawa couplings, and so on. Experimental bounds on these parameters will typically map out some finite size region in the space in which the flux lattice lives. Hence Ajtai's conjecture implies that finding BP flux vacua satisfying these constraints will in general be an NP-hard problem.

3.7 F-theory flux vacua and other combinatorial problems in string theory

The Bousso-Polchinski model's main physical weakness is that it ignores all moduli dependence of the potential. From a computational point of view, it does not explain the origin of the parameters g_{ij} of the lattice, which are in fact not free parameters in string theory. One might wonder if the actual instances of BP which arise are simpler than the worst case we discussed.

The next step in doing better is to construct IIB superstring flux vacua, or more generally F-theory flux vacua [41, 68]. Classically, one gets in this setting a discretuum of supersymmetric vacua with all complex structure (or shape) moduli stabilized, but the Kähler (or size) moduli unaffected. As pointed out in [78], taking into account quantum effects can supersymmetrically stabilize the Kähler moduli as well, leading to vacua with negative cosmological constant and no massless scalars. This was subsequently confirmed in examples in [45, 47], and extended to nonsupersymmetric vacua with negative cosmological constant and exponentially large compactification volume in [16, 39, 17]. A plausible construction to uplift these vacua to positive cosmological constant values was also proposed in [78], and variants thereof in [33, 99, 46].

A good zeroth order approximation to the study of the landscape of such flux vacua is to ignore the Kähler moduli altogether, and consider the vacua of the potential on the complex structure moduli space only, along the lines of [15, 44, 46].

To sketch the actual problem that arises in this fully string theoretic problem, and to show its relation to the idealized BP model, let us introduce some formalism (this is not important for subsequent sections however). F-theory flux is given by a harmonic 4-form G on an elliptically fibered Calabi-Yau 4-fold X . The flux G is uniquely determined by its

components with respect to an integral basis of harmonic 4-forms Σ_i , $i = 1, \dots, K$:

$$G = N^i \Sigma_i, \quad (3.8)$$

where $N^i \in \mathbb{Z}$, because of Dirac quantization. We can also add mobile D3-branes to the compactification. The four dimensional effective potential¹¹ induced by curvature, mobile D3-branes and flux is, in suitable units:

$$V = -\frac{\chi}{24} + N_{D3} + \frac{1}{2} \int_X G \wedge *G, \quad (3.9)$$

where χ is the Euler characteristic of X and N_{D3} the number of mobile D3-branes. Defining $\Lambda_0 \equiv -\chi/24 + N_{D3}$ and $g_{ij} \equiv \frac{1}{2} \int_X \Sigma_i \wedge * \Sigma_j$, and using (3.8), this becomes

$$V = \Lambda_0 + g_{ij} N^i N^j. \quad (3.10)$$

This is the same as the defining equation Eq. (3.1) of the BP model. However, the main difference is that the metric g_{ij} depends on an additional set of complex variables z^a , with $a = 1, \dots, h^{3,1}(X)$. Given a specific choice of N^i , their values are determined by minimizing the energy Eq. (3.1). This need only be a local minimum, so in general a choice of vacuum is now a choice of N^i and choice of minimum.

The source of this additional structure is that Ricci-flat metrics on a Calabi-Yau manifold come in continuous families, in part parameterized by the variables z^a (the complex structure moduli). Physically, such parameters lead to massless fields and long-range forces, which are typically in conflict with the data, so this is a problem. However, in the presence of flux, the potential energy Eq. (3.9) depends on the z^a , as the choice of CY metric enters into this expression through the Hodge star operator, so the standard principle that energy is minimized in a vacuum fixes these continuous variables and solves this problem (this is the major reason for the physical interest in this construction). While the presence of parameters such as the z^a is not obviously necessary to get a large vacuum multiplicity, it is true of all known models which have it.

There are various further subtleties not taken into account in the BP model. One is that the fluxes are constrained to satisfy the D3 charge cancellation condition,

$$-\frac{\chi}{24} + N_{D3} + \frac{1}{2} \int G \wedge G = 0. \quad (3.11)$$

Using this, one can rewrite Eq. (3.9) as

$$V = \frac{1}{4} \int_X (G - *G) \wedge *(G - *G). \quad (3.12)$$

This is still not very explicit, due to the presence of the $*$ -operator, but it can be shown that this equals

$$V_N(z) = e^{\mathcal{K}} (G^{A\bar{B}} D_A W \bar{D}_{\bar{B}} \bar{W} - 3|W|^2) \quad (3.13)$$

¹¹Consistent with our zeroth order approximation, we neglect an overall factor depending on the compactification volume (which is a Kähler modulus), and we neglect warping effects.

where [72]

$$W_N(z) = N^i \Pi_i(z), \quad \mathcal{K}(z, \bar{z}) = \Pi_i(z) Q^{ij} \bar{\Pi}_j(\bar{z}). \quad (3.14)$$

Here $\Pi_i(z) = \int_{\Sigma} \Omega(z)$ is the holomorphic period vector of the holomorphic 4-form, $G^{A\bar{B}}$ is the inverse metric on moduli space, D_A are compatible covariant derivatives, and $Q_{ij} \equiv \int_X \Sigma_i \wedge \Sigma_j$ is the intersection form on $H^4(X)$ and Q^{ij} its inverse. We omit further details, which can be found in [68, 78, 55, 44, 46] and the other references.¹²

The main point is that this part of the problem is mathematically precise and sufficiently concrete to make explicit computation of the potential $V_N(z)$ possible in examples for all choices of N and z . There are various approaches to computing the data we just described; for example the periods are determined by a Picard-Fuchs system of partial differential equations, which given a choice of Calabi-Yau manifold contains no adjustable parameters. In concrete constructions, the moduli at a minimum of V control observables such as coupling constants and masses of particles, and this model is perhaps the most fully realized example to date of how we believe string theory can in principle determine all the continuous parameters of the Standard Model from a starting point with no parameters, again always under the assumption that we know which vacuum to consider.

Given this setup, one can again ask for the existence of vacua with cosmological constant in a given small range. This problem is similar in spirit to the Bousso-Polchinski problem, but appears harder because of the coupling to the moduli, which will have different critical point values for each choice of flux. Therefore one would expect this problem to be at least as intractable, and no algorithm to exist that would guarantee a solution of the problem in polynomial time.

If we restrict attention to actual Calabi-Yau compactifications, then strictly speaking it does not really make sense to call this problem NP-hard, since NP-hardness is an asymptotic notion, and there are reasons to think that only a finite number of instances can actually be produced in string theory [56]. However, since the number of fluxes on elliptically fibered Calabi-Yau fourfolds can be at least as large as about 30,000 (and as large as about 2,000,000 for general Fermat fourfold hypersurfaces) [83], there is clearly little reason to doubt the effective intractability of the problem, at least in its current formulation.

Let us suggest a version of this problem with an asymptotic limit, in which the complexity question is well posed. As discussed for example in [55], one does not need an actual Calabi-Yau manifold X to pose this problem, merely a “prepotential,” a holomorphic function of $K/2 - 1$ complex variables z^i , which summarizes the geometric information which enters the problem. Thus, rather than input a lattice as in BP, we input a prepotential and a number χ as in Eq. (3.11), and ask whether the resulting set of flux vacua contains

¹²For those more familiar with this problem, the indices A, B range over both complex and Kähler moduli; the reintroduction of Kähler moduli at this level has as only effect to cancel off the negative contribution to V_N , in accord with the positive definite (3.12). However, after including quantum corrections [78], equation (3.12) no longer holds, while on general grounds (3.13) is still valid, but now the negative term is no longer cancelled off identically, and therefore the potential will not be positive definite. In fully stabilized models taking into account quantum corrections one can therefore expect (3.13) with A, B ranging over just the complex structure moduli to be a better model for the complex structure sector than the same omitting the negative term, as it would in the very special classical case.

one with V as defined in Eq. (3.13) in a specified range. Since we are not asking for the prepotential to correspond to an actual Calabi-Yau manifold, the problem size K can be arbitrarily large. One might also propose variations on this which capture more structure of the actual problem, such as to base the construction on variation of Hodge structure for some infinite set of manifolds.

Explicit algorithms to solve any of these problems, even approximately, would be of real value to string theorists. However, there is ample scope for reduction arguments which might prove their NP-completeness as well. For example, the Taylor series expansion of the prepotential contains far more information than a metric g_{ij} specifying a K -dimensional lattice, suggesting that a similar (though far more intricate) reduction argument could be made.

As discussed in the previous subsection, similar consideration hold for matching other continuous parameters, such as particle masses and Yukawa couplings. What about discrete quantities, such as gauge groups, numbers of generations and so on? Here again one encounters NP-hard problems: typically one needs to find all D-brane configurations in a given compactification consistent with tadpole cancellation and the discrete target requirements, which is very similar to subset sum problems. Such a combinatorial D-brane problem was studied in detail in a simple model in [28, 69]. In [69], it was suggested (without giving precise arguments) that this problem (in its asymptotic extension) is indeed NP-hard. Because of this, the authors had to resort to an exhaustive computer search of a large subset of solutions, scanning about 10^8 models, which required 4×10^5 CPU hours. To increase a certain measure of the problem size by a factor of 2, they estimated they would require a computation time of 10^8 CPU *years*. This makes the practical hardness of the problem quite clear.

Obviously, this suggestion would not mean that one cannot construct particular instances of models with, say, three generations. One might also hope to solve important pieces of the problem in polynomial time. But it would imply that one cannot construct general algorithms that systematically examine all solutions, finishing in guaranteed polynomial time (unless, of course, P turns out to equal NP).

We should also emphasize that NP-hardness does not mean that any instance of the problem will take exponential time to solve. NP-hardness is strictly speaking a worst case notion. Many instances may be easy to solve; for example we saw that the knapsack problem with a target range that is not too small compared to the typical size of the entries is effectively solvable in polynomial time.¹³ In general, when there is an exponentially large number of vacua satisfying the target constraints, finding one of them will be relatively easy. But such cases are of limited interest: physically, we want the constraints to be sufficiently tight to select only one or a few vacua, or at most select a manageable and enumerable set, and finding those tends to be exponentially hard for NP-hard problems. So we get a complementarity: predictivity versus computational complexity. The more selective a criterion, the harder it will be solve the associated selection problem.

¹³Actually, the well studied lattice problems such as SVP and CVP are not just worst case hard, but hard for average instances, a fact exploited in cryptography. It would be interesting to check whether BP shares this property.

However, when the selection criteria get so restrictive that one does not expect any solutions at all within a given ensemble (e.g. on statistical grounds), the problem of answering the question whether there are indeed no solutions may sometimes get much easier again. A trivial example is the subset sum problem for a target value close to the sum of all positive integers in the given list – one needs to check at most a few cases to solve this case. Thus, *excluding* certain ensembles of models may on general grounds still be a tractable task.

Finally, we note that there are often other, more efficient ways to extract physical predictions besides explicitly solving selection problems. There are plenty of complex physical systems for which finding the microscopic ground state or some set of microscopic states satisfying a number of macroscopic criteria is completely intractable, and yet one can learn a lot about the physics of these systems using statistical mechanics, even with only rough knowledge of the microscopic dynamics. Particularly relevant here is the theory of spin glasses, to which we return in section 4. Statistical methods to analyze the landscape of flux vacua were developed in [15, 44, 46], and we will discuss how to address this problem in that context in section 6.

3.8 Beyond toy models

Our discussion so far concerns toy models of the real string theory landscape, which are relatively rough approximations to the exact string theoretic problems. Even granting that these properly reflect the situation, or at least give a lower bound on the actual complexity of the landscape, we should discuss the added complications of more realistic problems. One obvious issue is the precision at which we can actually compute the cosmological constant and other parameters in a given string vacuum. If this is insufficient, we are in practice simply not in the position to even try to solve problems like those presented above.

We first note that there are string theoretic problems of the same general type we are discussing, in which it is known how to compute exact results, so that the discussion is precise. For example, we have the problem of finding a BPS black hole in IIB string theory on a Calabi-Yau threefold, whose entropy satisfies specified bounds. The entropy is determined by the charges N^i through the attractor mechanism [61]; it is the minimum of a function of the form $S = e^K |N^i \Pi_i|^2$. This problem is very similar to the F-theory flux vacua problem outlined above. However in the black hole case there are no approximations involved; everything is exact.

At present all such exactly solvable problems assume supersymmetry, and at least eight supercharges, so that one can get exact results both for the superpotential and Kähler potential. While one can hope for a similar level of control in $\mathcal{N} = 1$ supersymmetric vacua in the foreseeable future, it will be a long time before we have the computational ability to compute the cosmological constant to anything like 10^{-120} accuracy in even a single non-trivial example with broken supersymmetry. This will become clear after we outline how this is done in section 6.

If we were to grant that this will remain the permanent situation, then by definition the problem we are posing is intractable; no further arguments are required. However, there is no principle we know of that implies that this must remain so. Well controlled

series expansions or even exact solutions for many physical problems have been found, and although looking for one here may seem exceedingly optimistic from where we stand now, who can say what the theoretical situation will be in the fullness of time. Our point is rather that, granting that the large number of vacua we are talking about actually exist in the theory, presumably the natural outcome of such computations will be a list of numbers, the vacuum energies and other parameters in a large set of vacua, and that some sort of search of the type we are discussing would remain. Barring the discovery of extraordinary structure in the resulting list, the present evidence suggests that at this point one would run into a wall of computational complexity with its origins in deep theorems and conjectures of theoretical computer science, rather than just technical limitations.

4. Related computational problems in physics

There is a large literature exploring the relations between physics and computation, let us mention [27, 62, 49, 95]. What is of most relevance for us here is the particular case of finding the ground state(s) of complex physical systems. This has an even larger literature, both because of its physical and technological importance, and because it provides a particularly natural encoding of computational problems into physics.

4.1 Spin glasses

A prototypical example is the Sherrington-Kirkpatrick (SK) model of a spin glass [100]. A spin glass is a substance which contains dilute magnetic spins scattered about the sample, so that the individual spin-spin couplings vary in a quasi-random way. This can be modeled by a statistical mechanical system whose degrees of freedom are N two-valued spins $\sigma_i = \pm 1$, and the Hamiltonian

$$H = \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j. \quad (4.1)$$

In this model, the statistical sum over states is just the sum over spins; the partition function for a single sample is

$$Z = \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H} \quad (4.2)$$

at fixed J_{ij} . However, the couplings J_{ij} are not known *a priori*, as they vary from sample to sample.

Rather, we study an ensemble of spin glasses, specified by giving a probability distribution $[DJ]$ on the couplings. An expectation value in such an ensemble is defined as

$$\langle\langle X \rangle\rangle \equiv \int [DJ] \frac{1}{Z} \sum_{\{\sigma_i = \pm 1\}} e^{-\beta H} X.$$

where Z is as in Eq. (4.2). Although this expression averages over both the spins σ and the couplings J , they are treated differently – while the σ weights depend on J , the average over J is done with the dependence on σ removed. This is because these couplings were fixed during the preparation of the sample, long before any spin dynamics came into play. This is known as quenched disorder.

The SK model is defined by taking the couplings J_{ij} to be independent random variables, drawn from a Gaussian distribution with variance $1/N$. This normalization is made so that the free energy is $O(N)$ in the thermodynamic limit. Compared to a real spin glass, the main simplification of the SK model is that we have neglected the spatial distribution, in effect taking the limit of spins in infinite spatial dimensions.

Besides numerical study, some analytic information about such systems can be found by various tricks, the most general of which is the “replica trick” as discussed in [89]. The picture of the SK model which emerges is that, as a function of inverse temperature β , there are two regimes. At high temperature, the interactions are unimportant, and the spins are disordered. Conversely, at low temperature, there is ordering, but not detectable by a simple order parameter such as the average expectation value of the spins. It can be detected, for example, by the Edwards-Anderson order parameter

$$q_{EA} \equiv \frac{1}{N} \sum_{i=1}^N \langle \sigma_i \rangle^2$$

Furthermore, unlike an Ising model (the case with constant $J_{ij} < 0$), the partition function is not dominated by the lowest energy spin configuration, the global minimum of H . This is because of the existence of a large number of local minima (or “vacua”). These arise because of frustration.

Frustration refers to the fact that, for generic couplings J_{ij} , one will find many cases of $1 \leq i, j, k \leq N$ for which

$$J_{ij}J_{jk}J_{ki} < 0$$

and thus there is no preferred assignment of the spins σ_i , σ_j and σ_k which will minimize all three of their pairwise interaction energies. In this situation, stepwise lowering the energy of spin configurations tends to end in local minima, in which flipping one or a few spins raises the energy, but by flipping many spins one can reach a new potential well with a very different set of competing interactions and a different value of the potential at the minimum.

It is intuitively clear that the phenomenon of frustration makes the problem of finding the absolute minimal energy ground state for given couplings J_{ij} computationally hard, and indeed it has been shown that the problem is NP-hard [24].

Note the general similarity to the Bousso-Polchinski model. In particular, both models share the basic feature of possessing an exponentially large number of “vacua,” in BP because varying a single flux produces a large variation in energy, and in SK because of frustration.

As we mentioned earlier, this means that physical models typically will not find the global minimum within the relevant time scales, and this has real physical consequences for spin glasses. A spin glass magnetized at a high temperature in the presence of an external magnetic field shows a very characteristic relaxation behavior when cooled below its critical temperature and taken away from the external field. After a quick initial drop of magnetization to some characteristic value, it starts an extremely slow descent towards zero (or at least very small) magnetization. In fact it never quite reaches thermal equilibrium

within experimentally accessible time scales. The reason is that the system keeps on getting stuck in the exponentially large number of local metastable minima of its energy landscape, much like a computational relaxation algorithm would. As a result, the properties of such systems depend significantly on their preparation history and the waiting time: they are “aging.” This implies something rather remarkable, not shared by computationally simple systems such as ferromagnets or regular crystals, namely that just from observation of the system’s current state, we can infer some knowledge of its history.

4.2 Physically inspired methods for finding the ground state

Let us consider some ways in which one might try to find the ground state. The simplest is to choose a starting configuration and evolve by flipping single spins, accepting any which decrease the energy. This will rapidly stop at a local minimum. Physically, it corresponds to rapidly cooling the sample to zero temperature.

A more sophisticated approach which can deal with local minima is simulated annealing [80]. In this approach, we simulate the system according to the standard Metropolis algorithm for finite temperature. We recall that the Metropolis algorithm also proceeds by single spin flips, accepting any flip which lowers the energy, and accepting flips which raise the energy by ΔH with probability $\exp(-\beta\Delta H)$, thus generating the Boltzmann distribution.

Simulating annealing is then a Metropolis process which starts at high temperature, able to explore the entire configuration space, and then systematically lowers the temperature to zero. This process allows jumping out of local minima, while by taking the temperature to zero sufficiently slowly, one is confined to potential basins of lower and lower energy, eventually finding the global minimum. However, it has been argued [65] that to guarantee this requires taking $\beta \sim \log t$ at time t , *i.e.* the total process takes exponential time, just as would all known algorithms on a classical computer.

Again, these are worst case results, while in practice some problems can be solved. In these cases, clever algorithms can provide significant speedups. For example, spin glass techniques inspired the “cluster method” of Mézard et al [89], and many other methods such as deformation, basin hopping, and α branch and bound, have been found useful in practical optimization problems [110].

4.3 Reformulation of other NP-complete problems as physical systems

As mentioned earlier, it has been shown that the problem of finding the ground state of the spin glass Eq. (4.1) with specified couplings J_{ij} is NP-hard [24]. This was done by relating this to a graph theoretic problem which was known to be NP-hard.

A more natural result of this type is the mapping of the satisfiability problem into a spin glass with multi-spin interactions. This is very simple and is the starting point for a lot of work using statistical mechanical approaches to hard optimization problems.

Recall that in the 3-SAT problem, one is given a list of Boolean OR clauses, for example

$$x_2 \vee \bar{x}_7 \vee x_{22}$$

where the bar represents NOT, and \vee represents OR. The problem is to find an assignment of truth values to variables which satisfies all of the clauses.

This is trivially identical to the problem of finding a zero energy state of the following potential,

$$V = \sum_{i=1}^N \left(\frac{1 + J_{ia}\sigma_a}{2} \right) \left(\frac{1 + J_{ib}\sigma_b}{2} \right) \left(\frac{1 + J_{ic}\sigma_c}{2} \right)$$

where $\sigma_a = \pm 1$ represent the variables, and the coefficients $J_{ia}, J_{ib}, J_{ic} = \mp 1$ encode the choice of a variable or its negation in the i 'th clause. If any of $J_{ia}\sigma_a = -1$, the clause is satisfied and has zero energy, while if all $J_{ia}\sigma_a = +1$, the clause is not satisfied and has positive energy.

Various interesting insights were obtained by using this mapping as well as the general relations between optimization and statistical mechanics. As an example which inspired much interest [81], if we consider a randomly chosen 3-SAT problem (with randomly chosen J_{ia}) and vary the ratio α between the number of clauses and the number of variables, there appears to be a phase transition: for $\alpha < \alpha_c \sim 4.267$ the problems are satisfiable (meaning that as $N \rightarrow \infty$ the fraction of satisfiable problems goes to 1), while for $\alpha > \alpha_c$ they are not.

This phase transition is directly related to the difficulty of the problem. Finding an explicit solution is easy for small α , and becomes more difficult as constraints are added. Conversely, the difficulty of showing that no solution exists (a very different problem, as we discuss later) decreases with α . In some sense, the overall difficulty of the problem peaks at $\alpha = \alpha_c$.

4.4 Protein landscapes

Another much studied example is the potential landscape provided by configurations of proteins. The problem of finding the folded ground state of a protein (modeled by various discretized models) is known to be NP-hard [107], and simulations of protein folding based on these models suffer from the usual problem of getting stuck in metastable energy minima, making the problem computationally intractable already for relatively short sequences of amino acids. Again, the hardness of the problem has physical implications. Artificially made random sequences of amino acids generically do not fold properly: they do not find a unique folded ground state, as one would expect based on the NP-hardness of the problem. However, the story is quite different for biologically occurring proteins, which typically fold into a unique preferred ground state very quickly after being produced in a cell. These native states tend to be very stable, and proteins that are denaturated (that is, unfolded) by heating or chemical reactions often have no trouble folding back into their native state. Given the apparent computational complexity of the problem, this presents a puzzle, referred to as *Levinthal's paradox* [88].

The resolution of this paradox is evolution: the processes involved in synthesizing proteins, and in particular the actual amino acid sequence itself [104] have been selected over billions of years and a huge number of trials to be exactly such that biological folding is efficient and reliable. The particular landscape and folding pathways of natural proteins

are such that it is effectively funneled into a unique native state. Failure to do so would result in dysfunctional protein and weakening or elimination of the organism. In other words, whereas computational complexity is a notion based on worst case instances, there is strong evolutionary pressure to select for *best* case instances.¹⁴

In a way, the intractability of the general problem is again what allows the system to carry information about the past, in this case the whole process of evolution.

4.5 Quantum computation

It is interesting to ask whether using a quantum computer brings any speed-up in solving these problems, and this will be especially interesting for the second paper. For background on quantum computing, consult [82, 91].

Although there are many approaches to performing a computation using quantum mechanics, the most relevant for our discussion here is to translate it into the problem of finding a ground state of a quantum mechanical system, along the general lines we just discussed. The idea of doing computations this way gains particular interest from the concept of adiabatic quantum computing [58]. The idea is to find the ground state by starting from the known ground state of a simple Hamiltonian, and varying the Hamiltonian with time to one of interest. According to the adiabatic theorem of quantum mechanics, if this is done slowly enough, the original ground state will smoothly evolve into the ground state of the final Hamiltonian, and thus the solution of the problem. In fact any quantum computation can be translated into this framework, as recently shown by Aharonov *et al* [7].

Similar to the classical case, one defines complexity classes for problems solvable by quantum computers with specified resources. Of course for a quantum computer, all computation outputs are probabilistic, so all classes will have to specify the error tolerance as well. The class of problems viewed as tractable by a quantum computer is BQP, or Bounded-error Quantum Polynomial time. This is the class of decision problems solvable by a quantum computer in polynomial time, with at most $1/3$ probability of error. The number $1/3$ is conventional, any nonzero $\epsilon < 1/2$ would give an equivalent definition. If the probability of error is bounded in this way, one can always reduce the probability of error to an arbitrarily small value by repeating the computation a number of times, where the required number depends only on the desired accuracy. Many results relating classical and quantum complexity classes are known, see [38].

To admittedly oversimplify a complex and evolving story, while there are famous examples of problems for which quantum computers provide an exponential speedup, such as factoring integers [101], at present the evidence favors a simple hypothesis according to which a generic problem which takes time T for a classical computer, can be solved in time \sqrt{T} by a quantum computer.¹⁵ The simplest example of this is the Grover search algorithm

¹⁴But which instances these are depends on the details of the dynamics (in other words the algorithm), and *finding* these best case instances is conceivably again computationally hard. However, the mechanism of evolution provides enormous space and time resources to do this.

¹⁵As pointed out to us by Scott Aaronson, this hypothesis is simplistic, as there are also problems with no asymptotic speedup over classical computers, or with T^α speedup with $1/2 < \alpha < 1$. But perhaps it is

[70], and this result can be interpreted as providing general evidence for the hypothesis by the device of formulating a general computation as an oracle problem [27].

Many other cases of this type of speedup are known. Another relevant example is the problem of estimating an integral by probabilistic methods. As is well known, for a generic function with $O(1)$ derivatives, the standard Monte Carlo approach provides an estimate of the integral with $O(T^{-1/2})$ accuracy after sampling T points. If we assume a function evaluation takes unit time, this takes time T . On the other hand, a quantum computer can use T function evaluations to estimate the same integral to an accuracy T^{-1} . [71]

While significant, against exponential time complexity, a square root improvement does not help very much; an NP-hard problem will still take exponential time to solve. This also seems to come out of the adiabatic quantum computation framework, in which one constructs a family of Hamiltonians which adiabatically evolves to a Hamiltonian whose ground state solves an NP-hard problem. In the known examples, such a family of Hamiltonians will contain excited states with exponentially small gap above the ground state, so that the time required for adiabatic evolution is exponentially long (see [59] and references there).

The problems for which quantum computation is presently known to offer a more significant speedup are very special [102]. Many can be reformulated in terms of the “hidden subgroup problem,” which includes as a special case the problem of detecting periodicity in the sequence of successive powers of a number, exploited in Shor’s factoring algorithm. Of course lattice problems have an underlying abelian group structure as well and it is conceivable that quantum computers will turn out to have more power here.¹⁶

To conclude, it would be very interesting to have precise statements on the computational power of quantum field theory, compared to generic quantum mechanical systems. A precise discussion of this point would also enable us to discuss interesting questions such as whether computational power is invariant under duality equivalences [96]. It has been studied in depth for topological quantum field theory [63], but this is a rather special case, since for any given observable one can reduce such a theory to finitely many degrees of freedom. In contrast, formulating a general quantum field theory requires postulating an infinite number of degrees of freedom, the modes of the field at arbitrarily large energies. On the other hand, one expects that for any observable, there is some finite energy E , such that modes of larger energy decouple, and only finitely many modes enter in a non-trivial way. The question is to make this precise and estimate the number of quantum computational operations which are available as a function of physical resources time T , volume V and energy E .

Locality and dimensional analysis suggest that a general upper bound for the number of computations N which can be done by a $d + 1$ dimensional theory in time T and in a region of volume V should take the form $N \leq TVE^{d+1}$, where E has units of energy. However, it is not clear what determines E . The masses of the heaviest stable particles,

a reasonable first guess for the search problems which concern us here.

¹⁶A primary application of lattice algorithms is to cryptography, and we have been told that because of this, much of this research literature is government classified. For all we know, the technology we need to find string vacua may already exist at the NSA.

other natural scales in the theory, and properties of the initial conditions, might all play a role, and enter differently for different theories.

A closely related question is the difficulty of simulating a QFT by a quantum mechanical computer; *e.g.* what is the number of quantum gate operations required to compute the partition function or some other observable to a desired accuracy. The only directly relevant work we know of is [34], which suggests that simulating a lattice gauge theory with lattice spacing a requires TV/a^{d+1} computations, as one might expect. However, as defining a continuum QFT requires taking the limit $a \rightarrow 0$, this estimate is at best an intermediate step towards such a bound. One would need to use the renormalization group or similar physics to summarize all the dynamics at high energies by some finite computation, to complete this type of analysis.

5. Problems harder than NP

We now return to our string theoretic problems. One response to the difficulties of finding vacua with parameters such as Λ in a prescribed (e.g. by experiment) target range, as described in section 3, is to suggest that we have not taken into account all of the physics of early cosmology, and that properties of the initial conditions, dynamics or other effects will favor some vacuum over all others. Perhaps the problem of finding this “pre-selected” vacuum will turn out to be much easier than the problems described in section 3. All we would have to do then is to compute this preferred vacuum and compare to observations.

Here we consider a simple candidate principle which actually does this – in principle. As we will see in this section, trying to use it in practice leads to a computational problem more intractable than NP. We continue with a survey of additional concepts in complexity theory which will be useful in the sequel.

5.1 Sharp selection principles based on extremalization

What might be a principle which prefers or selects out a subset of vacua? From our present understanding of string theory, it seems unreasonable to hope for a principle which *a priori* selects out (say) a certain ten dimensional string theory, a particular Calabi-Yau threefold, bundle and fluxes, and so on. What seems more plausible is a principle that gives us an amplitude or probability distribution on the set of candidate vacua, which might be a function of their physical parameters, the size of the set of initial conditions which can evolve to the vacuum of interest, and so forth. This is usually referred to as a “measure factor” in the cosmological literature, and the probability distribution of vacua with these weights is the “prior distribution” or simply the “prior” (as in Bayesian analysis).

While one can imagine many possibilities, for definiteness let us consider, say, the idea that only vacua with positive cosmological constant can appear, and these with a probability which depends only on the c.c., as

$$P(\Lambda) \propto e^{24\pi^2 M_P^4/\Lambda}; \quad \Lambda > 0 \tag{5.1}$$

for positive Λ , and probability zero for $\Lambda \leq 0$. We grant that the sum of these factors over all metastable vacua is finite, so that this can be normalized to a probability distribution.

The exponent is the entropy $S(\Lambda)$ of four-dimensional de Sitter space with cosmological constant Λ [67].

This proposal has a long and checkered history which we will not try to recount (we will give more details in [48]). As, taken at face value, it appears to offer a solution to the cosmological constant problem, there are many works which have argued for and against it, perhaps the most famous being [76]. A simple argument for the proposal, using only general properties of quantum gravity, is that it follows if we grant that the number of microstates of a vacuum with cosmological constant Λ is proportional to $\exp S$, with S the dS entropy, and that these states undergo a dynamics involving transitions between vacua which satisfies the principle of detailed balance, as then this would be the expected probability of finding a statistical mechanical system in such a macrostate.

The proposal can be criticized on many grounds: the restriction to $\Lambda > 0$ is put in by hand, the relevant vacua in string theory are probably not eternal de Sitter vacua, and so on (see however [23] for recent, more detailed arguments in favor of this proposal). Furthermore, some specific frameworks leading to the proposal make other incorrect predictions. For example, the argument we mentioned might suggest that the resulting universe would be in a generic state of high entropy, predicting a cold and empty de Sitter universe.

In any case, if we simply take the proposal at face value, it at least makes a definite prediction which is not immediately falsified by the existing data, and thus it seems a good illustration of the general problem of using measure factors.

The measure factor Eq. (5.1) is extremely sharply peaked near zero, and thus for many distributions of Λ among physical vacua it is a good approximation to treat it as unity on the vacuum with the minimal positive cosmological constant Λ_{min} , and zero on the others. To illustrate this, let us grant that the distribution of cosmological constants near zero is roughly uniform, as is reasonable on general grounds [112], and as confirmed by detailed study [29, 46]. In this case, one expects the next-to-minimal value to be roughly $2\Lambda_{min}$, and the probability of obtaining this vacuum compared to that of the minimal vacuum is of order $\exp(-1/2\Lambda_{min})$, thus negligible. We will refer to the special case of a measure factor which is overwhelmingly peaked in this way as “pre-selection.”

We point out in passing that, to the extent that we believe that the cosmological evidence points to a specific non-zero cosmological constant of order $10^{-120} M_P^4$, there is a simple independent theoretical test of the proposal. It is that, now granting that the distribution of cosmological constants is roughly uniform over the entire range $(0, M_P^4)$, the total number of consistent metastable vacua should be approximately 10^{120} , since if it is much larger, we would expect the cosmological constant of the selected vacuum to be much smaller than the measured value (which would be a rather ironic outcome, given the history of this proposal). Granting this, the combination of these ideas would provide a simple explanation for the observed value of Λ , and in principle determine an overwhelmingly preferred unique candidate vacuum.

However, before we begin celebrating, let us now consider the problem of actually finding this preferred candidate vacuum. Given a concrete model such as BP, it is mathematically well posed; we simply need to find the minimum positive value attained by the c.c.. However, as one might imagine, proving that one has found a minimal value is more

difficult than simply finding a value which lies within a given range. Whereas the latter condition can be verified in polynomial time, here even verifying the condition would appear to require a search through all candidate vacua. Thus, apparently this problem is not even in NP.

To be more precise, we consider the decision problem *MIN-CC*, defined as the answer to the question,

Does the minimal positive c.c. of the theory lie in the range $[\Lambda_0 - \epsilon, \Lambda_0 + \epsilon]$?

Here Λ_0 could be the presently measured cosmological constant and ϵ the measurement error. To stay within the standard framework of complexity theory, we have formulated the problem as a decision problem, rather than as the problem of actually computing the minimal value. Note however that if we have an oracle that answers this question in one step, we can bracket the minimal value to a precision of order $1/2^n$ after n steps. The *MIN-CC* problem is equivalent to a positive answer to both of the following two decision problems:

1. *CC*: Does there exist a vacuum v with $0 < \Lambda(v) \leq \Lambda_0 + \epsilon$?
2. *MIN*: For all vacua w , does either $\Lambda(w) \leq 0$ or $\Lambda(w) > \Lambda_0 - \epsilon$?

While the first problem (*CC*) is in NP (assuming $\Lambda(v)$ can be computed in polynomial time, as is the case in the BP model), the second problem (*MIN*) appears of a different kind, since a positive answer to the question cannot be checked by a simple evaluation of $\Lambda(v)$ for some suitable v . In fact, it is by definition a problem in co-NP, the complementary class to NP: the problem “is X true?” is in co-NP iff the complementary problem “is X false?” is in NP.¹⁷ In this case, the complementary problem is

$$\exists w : 0 < \Lambda(w) \leq \Lambda_0 - \epsilon ?$$

which is clearly in NP. More generally, we have that NP problems have the logical structure $\exists w : R(\epsilon, w)$, while co-NP problems have the structure $\forall w : R(\epsilon, w)$, where R is some polynomial time predicate.

Thus, the problem *MIN-CC* is the conjunction of a problem in NP and a problem in co-NP. This is by definition in the class DP (for “Difference P”). An example of a universal (or complete) problem in this class is to decide if the *shortest* solution to the traveling salesman problem is of a given length l . While one can clearly solve this problem in finite time (by enumerating all solutions of length at most l), since it is not obviously in NP, it may be more difficult than problems in NP.

Complexity theorists strongly believe that the class DP is strictly larger (and thus intrinsically more difficult) than either NP or co-NP. As with $P \neq NP$, this belief is founded on experience with a large set of problems, and the consistency of a world-view formed from results which bear indirectly on the question. Of course, the underlying intuition,

¹⁷Recall there is an asymmetry between yes and no in the definition of NP: we only require a yes answer to be verifiable in polynomial time.

that finding an optimal solution should be harder than just finding a solution, is plausible and this might be enough for some readers. In the rest of this subsection we go on and briefly describe one of the main arguments for this, as explained in [98].

A standard way to think about such problems in complexity theory is to grant that we have an oracle which can solve some part of our problem in a single time step. For example, we might grant an oracle which, given a candidate v , answers the question MIN in one step. Given such an oracle, the problem $MIN-CC$ is in NP , as the remaining problem CC is in NP . Such a “relativized” class is denoted by superscripting with the class of the oracle, so the problem $MIN-CC$ is in the class NP^{co-NP} . This is much larger than DP , so at this point we have not learned much, but let us continue.

Now, an NP oracle can solve $co-NP$ problems, and vice versa. To see this, simply recall that by definition, the yes/no problem “is X true?” is in $co-NP$ iff the problem “is X false?” is in NP . A yes/no answer to the second question is also an answer to the first question, so NP and $co-NP$ oracles are the same.

Thus, NP^{co-NP} is the same as NP^{NP} , which is also called Σ_2 . This class answers questions of the form

$$\exists w_1 \forall w_2 R(\epsilon, w_1, w_2).$$

A physics example of this would be: “Is the height of the potential barrier between two given local minima x_i and x_f at least ϵ ?” Indeed this problem can be rephrased as “Does there exist a path γ from x_i to x_f such that for all points $x \in \gamma$ we have $V(x) - V(x_i) < \epsilon$?”, which (after some suitable discretization) fits the Σ_2 template. In other words, Σ_2 problems are decision versions of two step min-max problems. While there is no proof that these are more difficult than either NP or $co-NP$, one can continue anyways and iterate this construction, obtaining classes Σ_k which answer a question with a series of k alternating quantifiers. An example of such a question would be, given a two-player game (in which both players have perfect information, and the number of options is finite) and with a winner after k moves on each side, who has a winning strategy? Again, these are clearly finite problems, which would appear to become more and more difficult with increasing n .

The union of such problems defines the “polynomial hierarchy” PH of complexity classes (see [98], and also [109] for a short introduction and a physics study of such problems). Now its entire definition rests on the premise that $NP \neq co-NP$, so that existential quantification is different from universal quantification. Conversely, if the two are the same (as would be the case if $DP = NP$ or $co-NP$), this entire hierarchy would collapse to the simplest case of NP . While not disproven, this would lead to all sorts of counterintuitive claims that certain problems which seem much harder than others actually are not, which would be very surprising, leading to general acceptance of the premise $NP \neq co-NP$. The general style of argument shows an amusing resemblance to the generally accepted arguments for dualities between quantum field theories, string theories and so on, in theoretical physics (though here the point is the opposite, to argue that naively similar classes are in fact different).

The upshot of all this is that, while from the point of view of predictivity the measure factor Eq. (5.1) is very strong, in principle determining a unique candidate vacuum, using

it computationally is even more difficult than the NP hard problems we discussed earlier.

5.2 Even more difficult problems

For completeness, and to perhaps clear up some misconceptions, we should point out that there are even more difficult problems than the ones we considered. After P, a natural next deterministic class to define is EXP, the problems which can be solved in time which grows as an exponential of a polynomial of the problem size. One could instead restrict the available space. For example, PSPACE is the general class of problems which can be solved with storage space which grows at most as a polynomial in the problem size.

An easy and possibly relevant inclusion is $\text{PSPACE} \subseteq \text{EXP}$. Since a computer with N bits of storage space only has 2^N distinct states, this is the longest time it could possibly run without getting caught in a loop (one might call this the Poincaré recurrence time). Thus, all PSPACE problems can be solved in finite (though perhaps exponentially long) time.

We also have $\text{NP} \subseteq \text{PSPACE}$ (as are all the classes we discussed previously). To show this, we need to show that a program which generates all candidate solutions only needs polynomial space. This is easy to see for SAT, for example.

Of course, once one allows infinite sets into the discussion, one can have unsolvable problems, such as the Turing halting problem (decide whether a specified Turing machine halts on every input, or not). Unsolvable problems also arise in areas of mathematics which are closer to physics; perhaps the most relevant for string theory and quantum gravity is the following

Theorem For no compact n -manifold M with $n > 4$ is there an algorithm to decide whether another manifold M' is diffeomorphic to M .

(due to S. P. Novikov; see the references and discussion in [115], p. 73). Here can one imagine M' as given by some triangulation (of finite but unbounded size), or in any other concrete way. This follows by exploiting the unsolvability of the word problem for fundamental groups in $d > 4$.

It has been argued [66, 90] that this makes simple candidate definitions of the quantum gravity functional integral, for example as a sum over triangulations of a manifold M , uncomputable even in principle. While paradoxical, the idea is not in itself inconsistent; rather it would mean that such a physical model can in principle realize a larger class of computable functions than the original Church-Turing thesis. Indeed, if we believed in such a model, we might look for ways to make physical measurements which could extract this information, much as many now seek to build quantum computers to do computations more quickly than the classical model of computation allows.

While there is no evidence for this type of uncomputability in string theory, at present we seem far from having a complete enough formulation to properly judge this point. But there are interesting indirect consequences of these arguments for the structure of the landscape, as discussed for the geometry of Riemannian manifolds in [115, 90] and as we intend to discuss for the string theory landscape elsewhere.

5.3 Anthropic computing

We now take a step back on the complexity ladder. As we mentioned in section 2, one approach to vacuum selection is environmental selection, also known as the anthropic principle. Adding this ingredient clearly affects one’s expectations of the ability of cosmological dynamics to “compute” vacua with small cosmological constant or other particular properties. Our detailed discussion of cosmology appears in [48], but let us review here what kind of problems one could solve efficiently with a probabilistic computer when one allows for postselection on part of the output.

There are precise definitions of a complexity classes which allow for postselection. For quantum computers this is the class **PostBQP** (Bounded Quantum Polynomial time with Postselection) recently defined and studied by Aaronson [3]. These are the problems that can be solved in polynomial time using an “anthropic quantum computer”. The simplest and most colorful way to describe such a computer is that we give it an input, let it run, and postselect on the output satisfying some condition X , by killing ourselves if X is not true. **PostBQP** is then the class of problems that can be solved (probabilistically) in polynomial time by such a machine, *assuming* we survive. The difference with an ordinary quantum computer is thus that we are allowed to work with conditional probabilities instead of absolute probabilities.

The analogous class for classical probabilistic computers is **PostBPP**, which turns out to be equal to a class which was defined before computer scientists started thinking about the power of postselection, namely BPP_{path} [74, 38], and therefore this is the name usually used for this class. We will define BPP_{path} and explain why it equals **PostBPP** at the end of this subsection.

It is easy to see that **PostBQP** and **PostBPP** include **NP**. In fact these classes are larger than **NP**, but not unlimited; for example they are believed to be strictly smaller than **PSPACE** and **EXP**.

The formal definition of **PostBQP** is as follows. It consists of the languages L (a language is a particular set of N -bit strings for each N) of which membership can be determined as follows:

- We consider a quantum computer, in other words a unitary time evolution U acting on some Hilbert space \mathcal{H} , with U built out of a number of quantum gates (elementary unitary operations acting on a small number of qubits) which grows polynomially in the size N of the input. The Hilbert space \mathcal{H} has a tensor product decomposition

$$\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{V} \otimes \mathcal{W}$$

where $\mathcal{V} \cong \mathcal{W} \cong \mathbb{C}^2$, with basis $|0_{\mathcal{V}}\rangle$ and $|1_{\mathcal{V}}\rangle$ (resp. \mathcal{W}).

- A computation is defined as follows. We supply an input, a vector $v \in \mathcal{H}$ encoding the string x of which we want to decide whether it belongs to L , and receive an output Uv . We insist that the probability for measuring $|1_{\mathcal{V}}\rangle$ in Uv be nonzero for any input. The output is then the value of a measurement of the bit \mathcal{W} , conditioned on measuring $|1_{\mathcal{V}}\rangle$ in \mathcal{V} .

- We require probabilistic correctness, meaning that if $x \in L$, the output is $|1_{\mathcal{W}}\rangle$ with conditional probability at least $2/3$, and that if $x \notin L$, the output is $|0_{\mathcal{W}}\rangle$ with conditional probability at least $2/3$.

As in our definition of BQP in section 4.5, the precise number $2/3$ here is not significant as one can achieve a reliability arbitrarily close to 1 by repeating the computation.

In [3], it is proven that $\text{PostBQP} = \text{PP}$, in other words that the computations which can be performed this way are those in the class PP, a probabilistic but classical complexity class. The definition of PP is the class of problems which can be “solved” by a classical randomized computer (one with access to a random number source), in the sense that the output must be correct with probability greater than $1/2$.

This should be contrasted with another class, BPP, which is the class of problems which can be solved with probability of correctness and soundness greater than $2/3$. While these two classes may sound similar, they are vastly different, as it is generally believed that $\text{BPP} = \text{P}$ (and proven that it is contained in $\Sigma_2 \cap \Pi_2 \subseteq \text{PH}$ [87]), while PP is huge. The point is that, given an error probability p bounded **strictly** below $1/2$, one can run the same computation many times to achieve an exponentially small error probability, so BPP is almost as good as P for many purposes, and much used in real world computing.

On the other hand, since even flipping a coin has error probability $1/2$, having an error probability less than $1/2$, but no stricter bound, is not so impressive. A computer which produces a correct output for even the tiniest fraction of inputs, becoming negligible as the problem size increases, and otherwise flips a coin, would qualify as PP.

An example of a computation in PP, which is believed not to be in NP is: given a matrix M and integer k , is the permanent of M (defined like the determinant, but with all positive signs) greater than k ? Indeed, this problem is PP-complete, meaning it is not in NP unless $\text{NP} = \text{PP}$.

Despite its size, the class PP is believed to be smaller than PSPACE, not to mention EXP and larger classes. For example, it is not even clear that it contains PH, and there is an oracle relative to which it does not.

An example of a problem believed not to be in PP is the question of whether the game of Go has a winning strategy for one of the players, which (if we allow $n \times n$ boards) is in fact PSPACE-complete [93]. The difference between this and the simpler game theory problems we mentioned as being in PH is the length of the game, which is fixed in PH but can depend (polynomially) on the problem size here.

There are a number of surprising and suggestive equivalences of postselection classes with superficially different looking classes.

First, as we mentioned already in the beginning, $\text{PostBPP} = \text{BPP}_{\text{path}}$. The former is defined as the problems solvable on a probabilistic classical Turing machine in polynomial time with probability of error less than $1/3$, allowing for postselection on the output. A probabilistic classical Turing machine can also be thought of as a nondeterministic Turing machine in which $2/3$ of the paths accept if the answer is yes, and $2/3$ reject if the answer is no. In this representation, all computation paths must have equal length. The probabilistic interpretation is naturally obtained from this by choosing a random path, with each step

choice at a vertex in the tree having equal probability of being picked ($1/2$ if the paths split in two at each step). BPP_{path} is similarly defined [74, 38], but now without postselection, and instead allowing paths of different length (all polynomial). Probabilities can still be assigned proportional to the number of paths accepting or rejecting, but now this is not the same anymore as assigning stepwise equal probabilities.

That the two classes are equal can be seen as follows. First, PostBPP is contained in BPP_{path} . This is because, if we want to postselect on a property X , in BPP_{path} we can just create exponentially many copies of all computation paths for which property X is satisfied (by continuing the branching process), and not create copies of the paths where it is not, till the overwhelming majority of computation paths satisfy property X . This effectively postselects on X .

Second, BPP_{path} is contained in PostBPP . This is because, in the computation tree of a BPP_{path} machine, we can extend the shorter paths by a suitable number of branchings till all paths have equal length, labeling all but one of the new paths for each old path by a 0, and the other paths by a 1. Then in PostBPP , we can postselect on paths labeled 1.

Similar equivalences to classes that modify standard probability rules are true in the quantum case. PostBQP equals the class of problems that can be solved in polynomial time by a quantum computer with modified laws of quantum mechanics: either by allowing non-unitary time evolution (re-normalizing the total probability to 1 at the end), or by changing the measurement probability rule from $|\psi|^2$ to $|\psi|^p$ with $p \neq 2$ [3].

5.4 Advice

Postselection classes quantify the power of future boundary conditions. What about the power of past boundary conditions? This is quantified by the notion of *advice*. Classical advice is extra information $I(N)$ delivered to the Turing machine, depending only on the input size N , and not longer than a prescribed number $f(N)$ of bits. Thus one defines for example the class P/poly , the set of problems that can be solved in polynomial time with polynomial length advice. This is believed not to contain NP . It is also not contained in NP — in fact it even contains some undecidable problems. An example of advice would be a partial list of solutions of the problem for input length N . Note that for any decision problem, advice of length 2^N allows to solve the problem trivially, since we can simply give a list of all (yes/no) answers for all possible inputs of length N , which has length 2^N .

Quantum advice is defined similarly, but now the input can be thought of as some state described by $f(N)$ qubits. For example BQP/qpoly is the class of problems that can be solved by a quantum computer in polynomial with polynomial length quantum advice. Since the dimension of the Hilbert space spanned by $f(N)$ qubits is $2^{f(N)}$ dimensional, and could therefore in principle easily encode all solutions for all possible inputs, one might be tempted to conclude that this would be as powerful as exponential classical advice. However, there are very strong limitations on how much information can usefully be extracted from a quantum state, and indeed in [1] it is shown that $\text{NP} \not\subseteq \text{BQP/qpoly}$ relative to an oracle, and that $\text{BQP/qpoly} \subset \text{PP/poly}$, implying this class is not unlimited in scope. This supports the picture that an N -qubit quantum state is “more similar” to a probability distribution over N -bit strings than to a length 2^N string.

This ends our brief tour of complexity theory. Many of the ideas we introduced here will find interesting applications in part II.

6. Practical consequences

By “practical” we mean the question of how we as physicists trying to test string theory, or more generally to develop fundamental physics, should respond to these considerations. Of course the first response should be to focus on easy aspects of the problem, and avoid hard ones. While at present almost any problem one poses looks hard to do in generality, we believe there is a lot of scope for clever algorithms to enlarge the class of easy problems. But it is valuable to know beforehand when this is possible, and conversely to realize when a problem as presently formulated is intractable.

Since getting any handle on the set of candidate string vacua is so difficult, in [52] a statistical approach was set out, which has been pursued in [15, 44, 46, 39, 28, 55, 69] and elsewhere. A short recent overview is [85].

There is a fairly straightforward response to these issues in a statistical approach. It is to make the best use of what information and ability to compute the observables we do have. To do this, we should combine our information into a statistical measure of how likely we believe each candidate vacuum is to fit the data, including the cosmological constant other couplings, and discrete data. This can be done using standard ideas in statistics; let us outline how this might be done for the c.c., leaving details for subsequent work.

To try to prevent confusion at the start, we would certainly not advocate the idea that “our” vacuum must be the one which maximizes such a probability measure, which is clearly as much an expression of our theoretical ignorance as of the structure of the problem.¹⁸ Given additional assumptions, this might be an appropriate thing to do, or it might not. What one should be able to do is compare relative probabilities of vacua, always making clear the additional assumptions which entered into defining these.

Thus, we begin by imagining that we have a set of vacua V with index i , in which the cosmological constant is partially computable. (The same ideas would apply to a larger set of couplings, or other observables.) As a simple model, we might consider our set to be a class of string theory vacua, all of which realize the Standard Model at low energies, and with a classical (or “bare”) contribution to the cosmological constant modelled by the BP model. In other words, the data i specifying a vacuum is a vector of fluxes, and the classical cosmological constant Λ_0 is given by the formula Eq. (3.1).

Thus, our effective field theory is the Standard Model coupled to gravity, which we regard as defined at the cutoff scale $\mu \equiv 1 \text{ TeV}$. The observed value of the cosmological constant will then be a sum of the classical term and a series of quantum corrections, both perturbative and non-perturbative,

$$\Lambda = \Lambda_{bare} + g^2 F_2(\Lambda) + g^4 F_4(\Lambda) + \dots + e^{-F_{NP}(\Lambda)/g^2} + \dots \quad (6.1)$$

¹⁸The same comment of course applies to the vacuum counting measures introduced in [52]. There, the theoretical ignorance we were expressing was our lack of knowledge of what selects a vacuum; indeed a main point made there was that useful measures can be defined which do not assign probabilities to vacua at all.

The leading quantum correction F_2 will be given by a sum of one loop Feynman diagrams, and depends on all masses of particles, and other couplings. As this is an effective field theory, it involves an integral over a loop momentum $|p| \leq \mu$, the cutoff, so it is finite, but depends on μ as well. Finally, F_2 depends on the cosmological constant Λ as well, because the graviton propagator enters in the graviton one loop diagram. Now, we are most interested in finding vacua with very small Λ , and for this problem we can set $\Lambda = 0$ in this propagator and self-consistently impose $\Lambda = 0$ at the end. Similar arguments can be made for the higher order terms, and the final result is a constant shift Λ_{SM} to the quantity Λ_0 defined in Eq. (3.1).

Thus, quantum corrections due to the Standard Model do not modify the previous discussion in any qualitative way. However, to actually find the vacua with small Λ , we must know the quantity Λ_{SM} to a precision $10^{-60} \mu^4$ or so. Since the vacua with small Λ in the BP model, and all the other landscape models we know of, are widely spread through configuration space, even a tiny error here will spoil our ability to pose the problem, even leaving aside the later complexity considerations.

On general grounds, a perturbative series expansion such as Eq. (6.1) is an expansion in the marginal couplings $\alpha_i/2\pi$, where α_i include the gauge couplings $g_i^2/4\pi$ in each of the three gauge groups, as well as the Yukawa couplings. These range from order 1 for the top quark Yukawa, through 1/20 or so for QCD at 1TeV, down to 1/1000 or so for the electroweak $U(1)$. The QCD and top quark contributions are particularly problematic, as these series are asymptotic with typical maximal accuracy obtained by truncating the series after about $1/\alpha$ terms, in other words $\Delta\Lambda \sim (\alpha/2\pi)^{1/\alpha}$, so the desired accuracy is unattainable. Solving this problem and doing a reliable computation requires a non-perturbative framework, such as lattice gauge theory. Even before we reach this point, since the number of diagrams at a given loop order grows factorially, we encounter what may be intractable computational difficulties.

In contrast to the BP model and the stringy landscape, we will not claim that we know that this problem is intractable. It clearly has a great deal of structure, and we know of no reason in principle that a clever algorithm could not exist to compute the single number Λ_{SM} (given some precise definition for it) to arbitrary precision. On the other hand, it is clearly formidable. For the foreseeable future, one can only expect precise statements at leading orders, with hard work required to extend them to each subsequent order.

This might not sound like a reasonable physical problem to work on. We would agree, but nevertheless, let us consider the problem of using the data at hand, say the first one or two orders of the series expansion Eq. (6.1), along with some lattice gauge theory results, to improve our estimate of how likely a given vacuum is to describe our universe. What we would need to do first is derive a probability distribution

$$P(i, \Lambda)$$

which expresses the likelihood that vacuum i (in the toy model, i is a list of fluxes), has cosmological constant Λ . We start by taking the reliable results, the first orders of Eq. (6.1) and the (by assumption) exact data from the BP model, as the center of the distribution, call this $\Lambda_{BP} + \Lambda_{SM \text{ approx}}$. We then need to get an error estimate for the next order, and

make some hypothesis about how this error is likely to be distributed. Say this is Gaussian; we come up with the distribution

$$P(i, \Lambda) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-(\Lambda - \Lambda_{BP} - \Lambda_{SM \text{ approx}})^2 / 2\sigma^2}$$

where $\sigma^2 \propto g^4$ is the variance, estimated by the size of the first correction we dropped.

Obviously estimating Λ in a real string theory vacuum would be far more complicated, but the definition of the single vacuum distributions $P(i, \Lambda)$ should be clear. If we can compute them, what should we do with them? This depends on other assumptions; in particular the assumption of a prior measure factor on the vacua.

For definiteness, let us consider our standard $\exp c/\Lambda$ measure factor. In this case, we need to decide which vacuum realizes the minimum positive value of Λ . Of course, we cannot literally do this given the data at hand, but what we can do is find a probability distribution

$$P_{MIN-CC}(i)$$

which gives the probability with which the vacuum i would realize this minimum positive value, if the distributions $P(i, \Lambda)$ were accurate.

If we strictly follow the definition Eq. (6.1), the SM contribution will give a constant shift to all the vacuum energies, so the energies Λ_i of different vacua are highly correlated. Since the various choices of flux and configuration will affect the vacuum energy in the hidden sector as well, this is probably not very accurate; we would suspect that taking the individual Λ_i as independent random variables is likely to be a better model of the real string theory ensemble. In any case, let us first assume independence for simplicity, and then return to the original ensemble. Since the distributions $P(i, \Lambda)$ are smooth near zero, a good approximation to P_{MIN-CC} would simply be

$$P_{MIN-CC}(i) = \frac{P(i, 0)}{\sum_j P(j, 0)}. \quad (6.2)$$

Naively, the way that vacuum i can achieve the minimum is for it to realize $\Lambda = \epsilon > 0$ for some extremely small value of ϵ . Since the distributions are smooth, we can simply take $\Lambda = 0$ in evaluating the distribution. There is then a factor for the expected width of the Λ range over which vacuum i really is the minimum, but given independence this factor is the same for all vacua, and cancels out. We then normalize the resulting distribution to obtain Eq. (6.2).

It is not any harder to do this for the actual assumptions we made in our previous discussion, according to which there is a constant shift Λ_{SM} to the cosmological constant for all vacua, independent of the choice of vacuum. Now it is more efficient to think of the SM computations as providing a probability distribution

$$P_{SM}(\Lambda_{SM}) = \frac{1}{(2\pi)^{1/2}\sigma} e^{-(\Lambda_{SM} - \Lambda_{SM \text{ approx}})^2 / 2\sigma^2}.$$

The resulting probability distribution for vacua is that each vacuum with cosmological constant Λ appear with equal weight, the probability that the SM really did produce the

needed shift of the c.c. to give it a near zero value. The only difference between this and the previous discussion is the width factor, which is the difference between the i 'th c.c. and the next higher c.c. in the discretuum,

$$\Delta\Lambda_{BP}(i) = \Lambda_{BP}(i') - \Lambda_{BP}(i).$$

This leads to

$$P_{MIN-CC}(i) = \frac{\Delta\Lambda_{BP}(i)P_{SM}(-\Lambda_{BP}(i))}{\sum_j \Delta\Lambda_{BP}(j)P_{SM}(-\Lambda_{BP}(j))} \quad (6.3)$$

This is clearly much harder to compute than Eq. (6.2). In practice, independence between the c.c.'s in different vacua is probably a more realistic assumption, and we see that in fact this helps us.

An interesting point about this is that the original *MIN-CC* distribution did not need an additional prior (it was the prior) and thus led to a definite prediction (the minimum c.c.). By combining this with our theoretical ignorance, we obtain another probability distribution, again without an explicit prior. Of course, this is because we are now postulating a precise model of our theoretical ignorance.

In principle, one could use similar ideas to deal with the difficulty of actually finding the flux vacua which realize the correct c.c., by replacing the computational problem of finding the vacua which work, with that of estimating the distribution of vacua which work. This is not very interesting in our toy model, as the choice of flux had no observable consequences, but might be interesting in a more complicated and realistic model in which these choices interact. How practical is this? While these issues are certainly not the limiting factor in our present explorations of the landscape, it is conceivable that this sort of computability issue might someday arise.

Another application of these ideas might be to justify simplifying our picture of the landscape. For example, there is an optimistic hypothesis, which would remove most of the difficulty associated with the c.c. It is that the detailed distribution of c.c.'s is to a very good approximation independent of the “interesting” choices, which enter into all the other observable quantities. This seems to us very plausible as the c.c. receives additive contributions from all sectors, including hidden sectors which cannot be directly observed. In this case, while it is important to know how many vacua realize a c.c. within range of the observed one (after adding the remaining corrections), we do not need really need to know which specific vacua match the c.c. to make predictions; indeed this information would not be very useful. We would still be in the position of not being able to literally find the candidate vacua, but this would be more of philosophical interest.

Such a picture might be used to justify a style of argument discussed in [54] and references there, which could lead to qualitative predictions if the number of vacua is not too large. It is perhaps best described by a hypothetical example, modelled after [12, 106, 53, 51].

Suppose we could show that string theory contained 10^{160} vacua with the property X (say that supersymmetry is observable at upcoming collider experiments), and which realize all known physics, except for the observed c.c.. Suppose further that they realize a uniform distribution of cosmological constants; then out of this set we would expect about

10^{40} to also reproduce the observed cosmological constant. Suppose furthermore that 10^{100} vacua with property \bar{X} work except for possibly the c.c.; out of this set we only expect the correct c.c. to come out if an additional 10^{-20} fine tuning is present in one of the vacua which comes close. Not having any reason to expect this, and having other vacua which work, we have reasonable grounds for predicting X , in the strong sense that observing \bar{X} would be evidence against string theory.

While we cannot presently make such arguments precise, their ingredients are not totally beyond existing techniques in string compactification, up to the point where one needs precise results on the distribution of the c.c. and couplings. The preceding “optimistic hypothesis” would allow bypassing this point. Do we believe in it? It seems fairly plausible for Standard Model observables, but is perhaps less obvious for other properties, for example the properties of the dark matter. Rather than simply assume it, one could use the ideas we just discussed to estimate the correlation between the cosmological constant and other observables, and verify or refute this independence hypothesis.

7. Conclusions

The question of trying to understand the computational complexity of finding candidate vacua of string theory has two senses, a practical one concerning our efforts as physicists to find these vacua, and a more conceptual one of whether early cosmology can be usefully thought of as having in some sense “found” the vacuum we live in, by some process with a definable computational complexity. We will address the second sense of the question in a companion paper [48], building on the discussion in section 5 here to make precise statements such as “a cosmological model which can reliably find the vacuum with the minimum positive cosmological constant is more powerful than a polynomial time quantum computer with anthropic postselection.”

As to the first sense, we argued that, at least in the various simplified models now used to describe the landscape containing vacua of string theory, the problems of finding vacua which agree with known data are in a class generally believed to be computationally intractable. This means that, unless we can exploit some structure specific to string theory, or unless $P = NP$, we cannot expect to find an algorithm to do this much faster than an exhaustive search through all vacua. Since according to the standard anthropic arguments we need at least 10^{120} vacua to fit the cosmological constant, such an exhaustive search is clearly infeasible. Similar statements apply to many (though not all) aspects of the problem, such as jointly fitting the various observed parameters of the Standard Model.

Our strongest statement applies to the Bousso-Polchinski model, which is so similar to well studied NP complete problems that we could apply standard arguments. This is of course a crude approximation to the real problem in string theory, but the known next steps towards making this more realistic do not seem to us likely to change the situation. A concrete model in which this claim could be tested is the problem discussed in subsection 3.7 of finding supersymmetric F-theory vacua with prescribed V .

We considered various ways out, such as the use of approximation methods, or of measure factors derived from cosmology. Now one can sometimes show that approximate

solutions to problems are tractable, and it would be interesting to know if finding a vacuum in the BP model with cosmological constant $|\Lambda| < c/N_{vac}$ is tractable for some c . On the other hand, clearly if many vacua fit the existing data, we then face the possibility that they go on to make different predictions, and the theory becomes more difficult to test and falsify. So, while approximation methods are clearly important, this sort of “complementarity” means that we should be careful what we wish for.

To illustrate the situation with measure factors, we considered one popular candidate, the measure $\exp c/\Lambda$ depending only on the cosmological constant. This overwhelmingly favors the vacuum with the minimum positive cosmological constant, and from a theoretical point of view makes as definite a prediction as one could possibly hope for. But from a computational point of view, it is far more intractable than the mere problem of finding vacua which fit the data.

One can still hope of course that better understanding or a drastic reformulation of these problems will change the situation. It is important to remember that, if the number of candidate string vacua is finite, the problem of finding them is strictly speaking **not** NP-hard or in any other complexity class, as these are asymptotic concepts which describe the increase in difficulty as we consider larger problems in some class. We are merely reasoning from properties of the general (or even worst) case in families of problems, to guess at the difficulty of the specific case of interest. This type of reasoning is good for producing upper bounds but is not conclusive. Of course, it would be very interesting to have concrete proposals for how the type of difficulty we described could be avoided. We might even suggest the converse hope that, if a proposed solution does not entirely depend on specifics of the string theory problem, it could lead to new computational models or methods of general applicability.

Even if these difficulties turn out to be unavoidable, this need not imply that string theory is not testable. While at present it is not clear what experiment might prove decisive, there have been many proposed tests and even claims of observations that would pose great difficulties for the theory (an example, as discussed in [19], is a time-varying fine structure constant; the evidence is reviewed in [108]). One can certainly imagine finding direct evidence for or against the theory. In the near term, experiments to start in 2007 at the Large Hadron Collider at CERN, continuation of the spectacular progress in observational cosmology, and perhaps surprises from other directions, are likely to be crucial. If positive, such evidence might convince us that string theory is correct, while the problem of actually finding the single vacuum configuration which describes our universe remains intractable.

This raises the possibility that we might someday convince ourselves that string theory contains candidate vacua which could describe our universe, but that we will never be able to explicitly characterize them. This would put physicists in a strange position, loosely analogous to that faced by mathematicians after Gödel’s work. But it is far too early to debate just what that position might be, and we repeat that our purpose here is simply to extrapolate the present evidence in an attempt to make preliminary statements which could guide future work on these questions.

Acknowledgements

We acknowledge valuable discussions and correspondence with Scott Aaronson, Sanjeev Arora, Tom Banks, Raphael Bousso, Wim van Dam, Michael Freedman, Alan Guth, Jim Hartle, Shamit Kachru, Sampath Kannan, Sanjeev Khanna, Greg Kuperberg, Andrei Linde, Alex Nabutovsky, Dieter Van Melkebeek, Joe Polchinski, Eva Silverstein, Paul Steinhardt, Wati Taylor, Alex Vilenkin and Stephen Weeks. We particularly thank Scott Aaronson and Wati Taylor for a critical reading of the manuscript. MRD would like to thank John Preskill and Gerald Jay Sussman for helping him follow many of these topics over the years, Dave McAllester for introducing him to the computational aspects of the SK model, and Jean Gallier and Max Mintz for an inspiring visit to the UPenn CIS department.

This research was supported in part by DOE grant DE-FG02-96ER40959.

A. A simple pseudo-polynomial toy landscape solving algorithm

Here we give a simple algorithm for solving the toy landscape problem of section 3: are there any $m \in \{0, 1\}^N$ such that

$$\Lambda_0 - V_{\min} \leq \sum_{i=1}^N m_i \Delta V_i \leq \Lambda_0 - V_{\min} + \epsilon, \quad (\text{A.1})$$

where $\Delta V_i > 0$? We assume the V_i^\pm are known to a precision of order δ . The problem we are considering is then only sensible of course if $\delta < \epsilon/N$. Since there is an order δ uncertainty anyway, we are allowed to make rounding errors of order δ in each term of the sum. Hence, choosing energy units in which $\delta \equiv 1$, we can round off all ΔV_i to their closest integers values (as well as Λ_0 and V_{\min}), and work further over the integers.

Define for $K, s \in \mathbb{Z}^+$ the Boolean function $Q(K, s)$ to be **true** iff there is an $m \in \{0, 1\}^K$ such that

$$\sum_{i=1}^K m_i \Delta V_i = s. \quad (\text{A.2})$$

What we are eventually interested in is $Q(N, s)$ for s in the range $\Lambda_0 - V_{\min} \leq s \leq \Lambda_0 - V_{\min} + \epsilon$, or more precisely this interval extended by an amount of order $N\delta$ on both sides, since $N\delta$ is the maximal error of the sum. If $Q(N, s) = \text{false}$ in the entire range of this extended interval, we know the answer to our question is negative. If $Q(N, s) = \text{true}$ for at least one s well inside the original interval, where ‘well inside’ means more than order $N\delta$ away from the boundary, we know the answer to our question is positive. If $Q(N, s)$ happens to be **true** only near (i.e. within $N\delta$ of) the boundary of the original interval, we strictly speaking do not know the answer, since it depends now on the rounding errors.

The algorithm to compute Q is very simple. Note that trivially $Q(K, s) = \text{false}$ if $s < 0$ or $s > s_{\max} \equiv \sum_{i=1}^N \Delta V_i$, so one only needs to compute an $N \times s_{\max}$ matrix. Furthermore, the following recursion formula computes Q :

$$Q(K, s) = Q(K - 1, s) \text{ or } Q(K - 1, s - \Delta V_K), \quad (\text{A.3})$$

together with the initial condition $Q(0, s) = \text{true}$ iff $s = 0$. Thus this algorithm computes the required $Q(N, s)$ in

$$O(Ns_{max}) = O(N \sum_i \Delta V_i / \delta)$$

steps, where in the last expression we undid our choice of energy units $\delta \equiv 1$.

B. NP-completeness of subset sum

Here we show that the subset sum problem is NP-complete, by reducing the 3-SAT problem introduced in section 3.3 to it. The proof is standard, but we give it here for completeness, and to illustrate the sort of reasoning commonly used in such proofs. The version we present comes from [111].

The general 3-SAT problem has m clauses c_a , $a = 1, \dots, m$, each consisting of the disjunction of 3 boolean variables or their negation, chosen from a set of n boolean variables σ_i , $i = 1, \dots, n$. The question is if an assignment of truth values to the variables σ_i exists such that all clauses c_a are satisfied. A simple example of a $(m, n) = (3, 4)$ problem instance is the following:

$$\exists (\sigma_1, \sigma_2, \sigma_3, \sigma_4) : (\sigma_1 \vee \bar{\sigma}_2 \vee \sigma_3) \wedge (\bar{\sigma}_1 \vee \sigma_2 \vee \bar{\sigma}_4) \wedge (\bar{\sigma}_1 \vee \bar{\sigma}_2 \vee \bar{\sigma}_3) ?$$

We now give a polynomial reduction from 3-SAT to subset sum. Let c_{ai} be equal to 1 if σ_i appears (un-negated) in clause c_a , and equal to 0 otherwise. Similarly, let \bar{c}_{ai} be 1 if the negation $\bar{\sigma}_i$ appears in c_a , and 0 otherwise. Note that $\sum_i c_{ai} + \bar{c}_{ai} = 3$, because each clause contains 3 literals. Define $2n + 2m$ integers $\{x_i, \bar{x}_i, u_a, v_a\}$ in digital representation as follows:

$$\begin{aligned} x_i &= 10^{i-1} + \sum_{a=1}^m c_{ai} 10^{n+a-1} \\ \bar{x}_i &= 10^{i-1} + \sum_{a=1}^m \bar{c}_{ai} 10^{n+a-1} \\ u_a &= v_a = 10^{n+a-1} \end{aligned}$$

and an integer

$$t = \sum_{i=1}^n 10^{i-1} + \sum_{a=1}^m 3 \cdot 10^{n+a-1}.$$

For the example given above, this would be (with digits running down):

	x_1	x_2	x_3	x_4	\bar{x}_1	\bar{x}_2	\bar{x}_3	\bar{x}_4	u_1	u_2	u_3	v_1	v_2	v_3	t
c_3	0	0	0	0	1	1	1	0	0	0	1	0	0	1	3
c_2	0	1	0	0	1	0	0	1	0	1	0	0	1	0	3
c_1	1	0	1	0	0	1	0	0	1	0	0	1	0	0	3
σ_4	0	0	0	1	0	0	0	1	0	0	0	0	0	0	1
σ_3	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1
σ_2	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1
σ_1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	1

We claim that the 3-SAT problem we started from is equivalent to the subset sum problem

$$\exists k_i, \bar{k}_i, r_a, s_a \in \{0, 1\} : \sum_i k_i x_i + \bar{k}_i \bar{x}_i + \sum_a r_a u_a + s_a v_a = t ?$$

Indeed, this equation is

$$\sum_i (k_i + \bar{k}_i) 10^{i-1} + \sum_a \left[\left(\sum_i k_i c_{ai} + \bar{k}_i \bar{c}_{ai} \right) + r_a + s_a \right] 10^{n+a-1} = \sum_i 1 \cdot 10^{i-1} + \sum_a 3 \cdot 10^{n+a-1}.$$

Because each coefficient of the powers of 10 that appear lies between 0 and 5, this equality has to hold digit by digit. For the first n digits, this gives

$$\forall i : k_i + \bar{k}_i = 1,$$

which for any choice of $k_i \in \{0, 1\}$ is satisfied by taking $\bar{k}_i = 1 - k_i$. The last m digits then result in

$$\forall a : \left(\sum_i k_i c_{ai} + (1 - k_i) \bar{c}_{ai} \right) + r_a + s_a = 3,$$

which can be satisfied iff there is a choice of k_i such that $\forall a : \sum_i k_i c_{ai} + (1 - k_i) \bar{c}_{ai} \in \{1, 2, 3\}$. This is equivalent to the original 3-SAT problem, with identification $\sigma_i = \text{true}$ if $k_i = 1$ and $\sigma_i = \text{false}$ if $k_i = 0$, and with the value of the latter sum equal to the number of satisfied literals in clause c_a .

C. NP-completeness of 0/1-promise version of subset sum

Here we show that the 0/1-promise version of subset sum introduced in section 3.5 is NP-complete. We recall that this problem is the following. The input is a list of K positive integers y_a , $a = 1, \dots, K$ and a positive integer s , with the promise that for any choice of $m_a \in \mathbb{Z}^+$

$$\sum_a m_a y_a = s \quad \Rightarrow \quad m_a \in \{0, 1\}.$$

The question is if there *exists* a choice of $m_a \in \{0, 1\}$ such that $\sum_a m_a y_a = s$. We show this problem is NP-complete by polynomially reducing the standard subset sum problem to it.

The input of standard subset sum is a list of N positive¹⁹ integers x_i , $i = 1, \dots, N$ and an integer t . The question is if there exist $k_i \in \{0, 1\}$ such that $\sum_i k_i x_i = t$. We can assume that $0 < t < \sum_i x_i$, since otherwise the problem is trivial.

We reduce this to the 0/1-promise version as follows. Let

$$u = 2N \sum_i x_i$$

¹⁹The restriction to positive integers keeps the problem NP-complete, as follows e.g. directly from the proof in appendix B.

and define a list of $2N$ positive integers $\{y_i, \bar{y}_i\}$ and an integer s as

$$y_i = u^{N+2} + u^i + x_i, \quad \bar{y}_i = u^{N+2} + u^i, \quad s = Nu^{N+2} + \sum_i u^i + t. \quad (\text{C.1})$$

We claim that the subset sum problem we started from is equivalent to

$$\exists k_i, \bar{k}_i \in \{0, 1\} : k_i y_i + \bar{k}_i \bar{y}_i = s \quad (\text{C.2})$$

and that this is an instance of the promise version of the problem, i.e. for any choice of $k_i, \bar{k}_i \in \mathbb{Z}^+$:

$$\sum_i k_i y_i + \bar{k}_i \bar{y}_i = s \quad \Rightarrow \quad k_i, \bar{k}_i \in \{0, 1\}. \quad (\text{C.3})$$

The first claim is easily verified by writing out (C.2) using (C.1):

$$\sum_i (k_i + \bar{k}_i) u^{N+2} + \sum_i (k_i + \bar{k}_i) u^i + \sum_i k_i x_i = Nu^{N+2} + \sum_i u^i + t. \quad (\text{C.4})$$

For $k_i, \bar{k}_i \in \{0, 1\}$ the coefficients of the powers of u are guaranteed to be less than u , hence the equality has to hold power by power (“digit by digit” in base u). Then for any choice of $k_i \in \{0, 1\}$, \bar{k}_i is uniquely fixed to $\bar{k}_i = 1 - k_i$, and (C.4) reduces to $\sum_i k_i x_i = t$, which is the subset sum problem we started from.

It is slightly less trivial to show that the promise (C.3) is also satisfied, because *a priori* the $k_i, \bar{k}_i \in \mathbb{Z}^+$ are unbounded and therefore we cannot immediately say that (C.4) has to be satisfied power by power. However, it can be seen that (C.4) has to be satisfied at the highest power of u , i.e. $\sum_i k_i + \bar{k}_i = N$, essentially because any deviation would produce a left hand side much too large or much too small to match the right hand side. More precisely:

- If $\sum_i k_i + \bar{k}_i \geq N + 1$, then $\text{LHS} \geq (N + 1)u^{N+2} > \text{RHS}$ because $t < u$, so the equality cannot be satisfied.
- If $\sum_i k_i + \bar{k}_i \leq N - 1$, then $\text{LHS} \leq (N - 1)(u^{N+2} + \sum_i u^i + \sum_i x_i) < (N - 1)(u^{N+2} + u^{N+1}) < (N - 1)u^{N+2} + u^{N+2} = Nu^{N+2} < \text{RHS}$, because $\sum_i x_i < u$ and $N - 1 < u$.

Thus, $\sum_i k_i + \bar{k}_i = N$, and since $N < u$, $t < u$ and $\sum_i k_i x_i \leq N \sum_i x_i < u$, we have that every coefficient of the powers of u in (C.4) is strictly smaller than u , so the equation has to hold power by power. In particular this implies for the i -th power that $k_i + \bar{k}_i = 1$, and therefore $k_i, \bar{k}_i \in \{0, 1\}$, which is what we had to prove.

Finally note that the reduction is indeed polynomial: the number of bits needed to describe the input of the derived 0/1-promise subset sum problem is polynomial in the number of bits needed to describe the input of the standard subset problem we started from (because the number of bits of u is polynomial), and the number of steps required to do the reduction is clearly polynomial.

This completes the proof.

References

- [1] S. Aaronson, “Limitations of Quantum Advice and One-Way Communication,” quant-ph/0402095.
- [2] S. Aaronson, “Limits on Efficient Computation in the Physical World,” arXiv:quant-ph/0412143.
- [3] S. Aaronson, “Quantum Computing, Postselection, and Probabilistic Polynomial-Time,” arXiv:quant-ph/0412187.
- [4] S. Aaronson, “NP-complete Problems and Physical Reality,” [arXiv:quant-ph/0502072].
- [5] L. F. Abbott, “A Mechanism For Reducing The Value Of The Cosmological Constant,” Phys.Lett. B150 (1985) 427.
- [6] V. Agrawal, S. M. Barr, J. F. Donoghue and D. Seckel, “The anthropic principle and the mass scale of the standard model,” Phys. Rev. D **57**, 5480 (1998) [arXiv:hep-ph/9707380].
- [7] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd and O. Regev, “Adiabatic Quantum Computation is Equivalent to Standard Quantum Computation,” [arXiv:quant-ph/0405098].
- [8] M. Ajtai, “The shortest vector problem in L2 is NP-hard for randomized reductions (extended abstract),” in: Proceedings of the thirtieth annual ACM symposium on Theory of computing, 1998, ACM, New York, 10-19.
- [9] M. Ajtai, “Random lattices and a conjectured 0-1 law about their polynomial time computable properties,” Foundations of Computer Science, 2002. Proceedings of the 43rd IEEE FOCS 2002. Revised version at <http://eccc.uni-trier.de/ecce-reports/2002/TR02-061/Paper.ps>.
- [10] M. Agrawal, N. Kayal and N. Saxena, “PRIMES is in P”, Annals of Mathematics, 160 (2004), 781-793.
- [11] I. Antoniadis, “The physics of extra dimensions,” [arXiv:hep-ph/0512182].
- [12] N. Arkani-Hamed and S. Dimopoulos, “Supersymmetric unification without low energy supersymmetry and signatures for fine-tuning at the LHC,” JHEP **0506**, 073 (2005) [arXiv:hep-th/0405159].
- [13] N. Arkani-Hamed, S. Dimopoulos and S. Kachru, “Predictive landscapes and new physics at a TeV,” [arXiv:hep-th/0501082].
- [14] S. Arora, *Computational Complexity: A Modern Approach*, <http://www.cs.princeton.edu/arora/book/book.html>
- [15] S. Ashok and M. R. Douglas, “Counting flux vacua,” JHEP **0401**, 060 (2004) [arXiv:hep-th/0307049].
- [16] V. Balasubramanian and P. Berglund, “Stringy corrections to Kaehler potentials, SUSY breaking, and the cosmological constant problem,” JHEP **0411**, 085 (2004) [arXiv:hep-th/0408054].
- [17] V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, “Systematics of moduli stabilisation in Calabi-Yau flux compactifications,” JHEP **0503**, 007 (2005) [arXiv:hep-th/0502058].

- [18] T. Banks, M. Dine and N. Seiberg, “Irrational axions as a solution of the strong CP problem in an eternal universe,” *Phys. Lett. B* **273**, 105 (1991) [arXiv:hep-th/9109040].
- [19] T. Banks, M. Dine and M. R. Douglas, “Time-varying alpha and particle physics,” *Phys. Rev. Lett.* **88**, 131301 (2002) [arXiv:hep-ph/0112059].
- [20] T. Banks, “Heretics of the false vacuum: Gravitational effects on and of vacuum decay. II,” arXiv:hep-th/0211160.
- [21] T. Banks, “Landskepticism or why effective potentials don’t count string models,” arXiv:hep-th/0412129.
- [22] T. Banks, “More thoughts on the quantum theory of stable de Sitter space,” arXiv:hep-th/0503066.
- [23] T. Banks and M. Johnson, “Regulating Eternal Inflation,” arXiv:hep-th/0512141.
- [24] F. Barahona, *J. Phys. A: Math. Gen.*, 15 (1982) 3241-3253.
- [25] J. D. Barrow and F. J. Tipler, “The Anthropic Cosmological Principle,” Oxford University Press, 1988.
- [26] B. Bates and F. Denef, “Exact solutions for supersymmetric stationary black hole composites,” arXiv:hep-th/0304094.
- [27] C. H. Bennett, “The Thermodynamics of Computation: a review,” *Int. J. Theor. Phys.* **21**, 12 (1982) 905–940.
- [28] R. Blumenhagen, F. Gmeiner, G. Honecker, D. Lust and T. Weigand, “The statistics of supersymmetric D-brane models,” *Nucl. Phys. B* **713**, 83 (2005) [arXiv:hep-th/0411173].
- [29] R. Bousso and J. Polchinski, “Quantization of four-form fluxes and dynamical neutralization of the cosmological constant,” *JHEP* **0006**, 006 (2000) [arXiv:hep-th/0004134].
- [30] J. D. Brown and C. Teitelboim, “Dynamical Neutralization Of The Cosmological Constant,” *Phys. Lett. B* **195** (1987) 177.
- [31] J. D. Brown and C. Teitelboim, “Neutralization Of The Cosmological Constant By Membrane Creation,” *Nucl. Phys. B* **297**, 787 (1988).
- [32] J.D. Bryngelson, J.N. Onuchic, N.D. Socci, P.G. Wolynes, “Funnels, Pathways and the Energy Landscape of Protein Folding: A Synthesis”, *Proteins-Struct. Func. and Genetics.* **21** (1995) 167, [arXiv:chem-ph/9411008]
- [33] C. P. Burgess, R. Kallosh and F. Quevedo, “de Sitter string vacua from supersymmetric D-terms,” *JHEP* **0310**, 056 (2003) [arXiv:hep-th/0309187].
- [34] T. Byrnes and Y. Yamamoto, “Simulating lattice gauge theories on a quantum computer,” [arXiv:quant-ph/0510027].
- [35] S. M. Carroll, “Why is the universe accelerating?,” eConf **C0307282**, TTH09 (2003) [AIP Conf. Proc. **743**, 16 (2005)] [arXiv:astro-ph/0310342].
- [36] http://www.claymath.org/millennium/P_vs_NP
- [37] S. R. Coleman and F. De Luccia, “Gravitational Effects On And Of Vacuum Decay,” *Phys. Rev. D* **21**, 3305 (1980).
- [38] The Complexity Zoo, http://qwiki.caltech.edu/wiki/Complexity_Zoo

- [39] J. P. Conlon and F. Quevedo, “On the explicit construction and statistics of Calabi-Yau flux vacua,” JHEP **0410**, 039 (2004) [arXiv:hep-th/0409215].
- [40] S. Cook, “The importance of the P versus NP question,” JACM **50**, 1 (2003) pp. 27–29.
- [41] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux,” JHEP **9908**, 023 (1999) [arXiv:hep-th/9908088].
- [42] F. Denef, “Supergravity flows and D-brane stability,” JHEP **0008**, 050 (2000) [arXiv:hep-th/0005049].
- [43] F. Denef, “Quantum quivers and Hall/hole halos,” JHEP **0210**, 023 (2002) [arXiv:hep-th/0206072].
- [44] F. Denef and M. R. Douglas, “Distributions of flux vacua,” JHEP **0405**, 072 (2004) [arXiv:hep-th/0404116].
- [45] F. Denef, M. R. Douglas and B. Florea, “Building a better racetrack,” JHEP **0406**, 034 (2004) [arXiv:hep-th/0404257].
- [46] F. Denef and M. R. Douglas, “Distributions of nonsupersymmetric flux vacua,” JHEP **0503** (2005) 061 [arXiv:hep-th/0411183].
- [47] F. Denef, M. R. Douglas, B. Florea, A. Grassi and S. Kachru, “Fixing all moduli in a simple F-theory compactification,” arXiv:hep-th/0503124.
- [48] F. Denef and M. R. Douglas, “Computational Complexity of the Landscape II: Cosmological Considerations,” to appear.
- [49] D. Deutsch, “Quantum theory, the Church-Turing Principle and the universal quantum computer,” Proc. R. Soc. Lond. A, 400:97, 1985.
- [50] B. S. DeWitt, “Quantum Theory Of Gravity. 1. The Canonical Theory,” Phys. Rev. **160**, 1113 (1967).
- [51] M. Dine, “Supersymmetry, naturalness and the landscape,” arXiv:hep-th/0410201.
- [52] M. R. Douglas, “The statistics of string / M theory vacua,” JHEP **0305**, 046 (2003) [arXiv:hep-th/0303194].
- [53] M. R. Douglas, “Statistical analysis of the supersymmetry breaking scale,” arXiv:hep-th/0405279.
- [54] M. R. Douglas, “Basic results in vacuum statistics,” Comptes Rendus Physique **5**, 965 (2004) [arXiv:hep-th/0409207].
- [55] M. R. Douglas, B. Shiffman and S. Zelditch, “Critical points and supersymmetric vacua. III: String/M models,” arXiv:math-ph/0506015.
- [56] M. R. Douglas and Z. Lu, “Finiteness of volume of moduli spaces,” arXiv:hep-th/0509224.
- [57] <http://physics.nist.gov/cuu/Constants/energy.html>
- [58] E. Farhi, J. Goldstone, S. Gutmann and M. Sipser, “Quantum Computation by Adiabatic Evolution,” [arXiv:quant-ph/0001106].
- [59] E. Farhi, J. Goldstone, S. Gutmann and D. Nagaj, “How to Make the Quantum Adiabatic Algorithm Fail,” [arXiv:quant-ph/0512159].

- [60] J. L. Feng, J. March-Russell, S. Sethi and F. Wilczek, “Saltatory relaxation of the cosmological constant,” Nucl. Phys. B **602**, 307 (2001) [arXiv:hep-th/0005276].
- [61] S. Ferrara, R. Kallosh and A. Strominger, “N=2 extremal black holes,” Phys. Rev. D **52** (1995) 5412 [arXiv:hep-th/9508072].
- [62] R. P. Feynman, “Simulating Physics with Computers,” Int. J. Theor. Phys. **21** 6/7 (1982), 467–488.1
- [63] M. H. Freedman, A. Kitaev and Z. Wang, Commun. Math. Phys. **227**, 587 (2002) [arXiv:quant-ph/0001071].
- [64] M.R. Garey and D.S. Johnson, “Computers and Intractability: A Guide to the Theory of NP-Completeness,” W. H. Freeman and Company (1979).
- [65] S. Geman and D. Geman, IEEE Trans. PAMI-6, 721 (1984).
- [66] R. Geroch and J. B. Hartle, “Computability and Physical Theories,” in *Between Quantum and Cosmos: Studies and Essays in Honor of John Archibald Wheeler*, eds. W. H. Zurek, A. van der Merwe and W. A. Miller, Princeton Univ. Press, 1988.
- [67] G. W. Gibbons and S. W. Hawking, “Action Integrals And Partition Functions In Quantum Gravity,” Phys. Rev. D **15**, 2752 (1977).
- [68] S. B. Giddings, S. Kachru and J. Polchinski, “Hierarchies from fluxes in string compactifications,” Phys. Rev. D **66**, 106006 (2002) [arXiv:hep-th/0105097].
- [69] F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lust and T. Weigand, “One in a billion: MSSM-like D-brane statistics,” arXiv:hep-th/0510170.
- [70] L. K. Grover, “A Fast Quantum Mechanical Algorithm for Database Search,” STOC 28 (1996) 212–219.
- [71] L. K. Grover, Proc. 30 Annual ACM Symp. on Theory of Computing, ACM Press; [arXiv:quant-ph/9711043].
- [72] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi-Yau four-folds,” Nucl. Phys. B **584**, 69 (2000) [Erratum-ibid. B **608**, 477 (2001)] [arXiv:hep-th/9906070].
- [73] A. H. Guth, “Inflation and eternal inflation,” Phys. Rept. **333**, 555 (2000) [arXiv:astro-ph/0002156].
- [74] Y. Han, L. Hemaspaandra and T. Thierauf, “Threshold computation and cryptographic security,” SIAM Journal on Computing 26(1):59-78, 1997.
- [75] J. B. Hartle and S. W. Hawking, “Wave Function Of The Universe,” Phys. Rev. D **28**, 2960 (1983).
- [76] S. W. Hawking, “The Cosmological Constant Is Probably Zero,” Phys. Lett. B **134**, 403 (1984).
- [77] D.S. Johnson, “The NP-completeness column,” ACM transactions on algorithms 1:1, 160-176 (2005).
- [78] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, “De Sitter vacua in string theory,” Phys. Rev. D **68**, 046005 (2003) [arXiv:hep-th/0301240].
- [79] R.M. Karp, “Reducibility Among Combinatorial Problems,” in Complexity of Computer Computations, Proc. Sympos. IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y.. New York: Plenum, p.85-103. 1972.

- [80] S. Kirkpatrick, C. D. Gelatt, Jr. and M. P. Vecchi, “Optimization by Simulated Annealing,” *Science* **220** (1983) 671–680.
- [81] S. Kirkpatrick and B. Selman, “Critical Behavior in the Satisfiability of Random Boolean Expressions,” *Science* **264** (1994) 1297–1301.
- [82] A. Yu. Kitaev, A. H. Shen, and M. N. Vyalyi, *Classical and Quantum Computation*, AMS 1992.
- [83] A. Klemm, B. Lian, S. S. Roan and S. T. Yau, “Calabi-Yau fourfolds for M- and F-theory compactifications,” *Nucl. Phys. B* **518**, 515 (1998) [arXiv:hep-th/9701023].
- [84] P.N. Klein and N.E. Young, “Approximation algorithms for NP-hard optimization problems”, *Algorithms and Theory of Computation Handbook*, CRC Press, 1999, <http://www.cs.brown.edu/people/pnk/publications/1998approxOpt.ps>
- [85] J. Kumar, “A review of distributions on the string landscape,” arXiv:hep-th/0601053.
- [86] A.K. Lenstra, H.W. Lenstra and L. Lovász, “Factoring polynomials with rational coefficients,” *Math. Ann.* **261**, 515–534.
- [87] C. Lautemann, “BPP and the polynomial time hierarchy,” *Inf. Proc. Lett.* **17** (1983) 215–218.
- [88] C. Levinthal, “How to fold graciously,” in *Mossbauer Spectroscopy in Biological Systems*, eds. P. DeBrunner *et al*, p. 22, U. of Illinois Press, Urbana, 1969.
- [89] M. Mézard, G. Parisi and M. A. Virasoro, *Spin Glass Theory and Beyond*, World Scientific, 1987.
- [90] A. Nabutovsky and R. Ben-Av, “Noncomputability arising in dynamical triangulation model of four-dimensional Quantum Gravity,” *Comm. Math. Phys.* **157** 1 (1993) 93–98.
- [91] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000.
- [92] For an updated online list of NP-complete optimization problems and known approximation results, see <http://www.nada.kth.se/~viggo/problemlist/compendium.html>
- [93] C. Papadimitriou, *Computational Complexity*, Addison-Wesley, Reading, MA, 1994.
- [94] J. Polchinski, “Introduction to cosmic F- and D-strings,” arXiv:hep-th/0412244.
- [95] J. Preskill, “Quantum information and physics: some future directions,” *J.Mod.Opt.* **47** (2000) 127–137, [arXiv:quant-ph/9904022].
- [96] J. Preskill, “Quantum information and the future of physics,” 2002 lecture.
- [97] V. A. Rubakov, “Large and infinite extra dimensions: An introduction,” *Phys. Usp.* **44**, 871 (2001) [*Usp. Fiz. Nauk* **171**, 913 (2001)] [arXiv:hep-ph/0104152].
- [98] *Computational Complexity Theory*, IAS/Park City Mathematics series vol. 10, S. Rudich and A. Wigderson, eds., AMS 2004.
- [99] A. Saltman and E. Silverstein, “The scaling of the no-scale potential and de Sitter model building,” *JHEP* **0411**, 066 (2004) [arXiv:hep-th/0402135].
- [100] D. Sherrington and S. Kirkpatrick, “Solvable Model of a Spin-Glass,” *Phys. Rev. Lett.* **35** (1975) , 1792–1796.

- [101] P. W. Shor, “Algorithms for Quantum Computation: Discrete Logarithms and Factoring,” FOCS 35 (1994) 124–134.
- [102] P. W. Shor, “Progress in Quantum Algorithms,” available at <http://www-math.mit.edu/~shor/elecpubs.html>.
- [103] L. Smolin, “How far are we from the quantum theory of gravity?,” arXiv:hep-th/0303185.
- [104] M. Socolich, S.W. Lockless, W.P. Russ, H. Lee, K.H. Gardner, R. Ranganathan “Evolutionary information for specifying a protein fold,” Nature. 2005 Sep 22; 437(7058):512-8.
- [105] L. Susskind, “The anthropic landscape of string theory,” [arXiv:hep-th/0302219].
- [106] L. Susskind, “Supersymmetry breaking in the anthropic landscape,” arXiv:hep-th/0405189.
- [107] R. Unger and J. Moult, “Finding the lowest free energy conformation of a protein is an NP-hard problem: proof and implications,” Bull Math Biol. 1993 Nov;55(6):1183-98. For a review see [32].
- [108] J. P. Uzan, “The fundamental constants and their variation: Observational status and Rev. Mod. Phys. **75**, 403 (2003) [arXiv:hep-ph/0205340].
- [109] P. Varga, Minimax Games, Spin Glasses and the Polynomial-Time Hierarchy of Complexity Classes, cond-mat/9604030.
- [110] D. J. Wales, *Energy Landscapes*, Cambridge Univ. Press, 2003.
- [111] I. Wegener, “Complexity Theory: Exploring the Limits of Efficient Algorithms,” Springer, 2004.
- [112] S. Weinberg, “Anthropic Bound On The Cosmological Constant,” Phys. Rev. Lett. **59**, 2607 (1987).
- [113] S. Weinberg, “The Cosmological Constant Problem,” Rev. Mod. Phys. **61**, 1 (1989).
- [114] S. Weinberg, “Einstein’s Mistakes,” Physics Today, November 2005, 31–35.
- [115] S. Weinberger, *Computers, Rigidity and Moduli*, Princeton Univ. Press, 2005.
- [116] J.A. Wheeler, “Superspace and the Nature of Quantum Geometro Dynamics,” in Battelle Rencontres 1967 p. 253, eds. C. DeWitt and J.A. Wheeler (Benjamin, New York 1968).
- [117] http://en.wikipedia.org/wiki/List_of_complexity_classes
- [118] S. Wright, “The roles of mutation, inbreeding, crossbreeding and selection in evolution,” *Proceedings of the Sixth International Congress on Genetics* **1** (1932) , 356–366.
- [119] A. C.-C. Yao, “Classical physics and the Church–Turing Thesis,” JACM **50**, 1 (2003) 100–105.